Sarkisov program for algebraically integrable and threefold foliations

Birational Geometry and Related Topics at XJTLU

Yanze Wang

AMSS

2025/12/25

Table of Contents

Table of Contents

Sarkisov, Reid and Corti

Origin of Sarkisov Program.

- Sarkisov introduced elementary links (Sarkisov links) and annonced that any birational map of 3-fold Mori fibre spaces is a composition of Sarkisov links.
- Reid reviewed Sarkisov's work and outlined some key ideas.
- Corti gave a proof.

Theorem (Corti95)

Let $\Phi : X \dashrightarrow Y$ be a birational map of $\mathbb{Q}$-factorial terminal threefolds, and let $f : X \to S$ and $g : Y \to T$ be two Mori fibre spaces. Then Φ is a composition of finitely many Sarkisov links $\phi_i : X_i \to X_{i+1}$.

Sarkisov links

Definition

A Sarkisov link between two Mori fibre spaces is one of following:

$$
\begin{tikzcd}
V_i \arrow[r, dashed, "\phi_i"] \arrow[d, "p_i"'] & X_{i+1} \arrow[d, "f_{i+1}"] \\
X_i \arrow[d, "f_i"'] & Z_{i+1} \arrow[dl, "\beta_{i+1}"] \\
Z_i &
\end{tikzcd}
$$

I

$$
\begin{tikzcd}
V_i \arrow[r, dashed, "\phi_i"] \arrow[d, "p_i"'] & V_{i+1} \arrow[d, "q_{i+1}"] \\
X_i \arrow[d, "f_i"'] & X_{i+1} \arrow[d, "f_{i+1}"] \\
Z_i \arrow[r, "\cong"] & Z_{i+1}
\end{tikzcd}
$$

II

$$
\begin{tikzcd}
X_i \arrow[r, dashed, "\phi_i"] \arrow[d, "f_i"'] & V_{i+1} \arrow[d, "q_{i+1}"] \\
Z_i \arrow[dr, "\alpha_i"'] & X_{i+1} \arrow[d, "f_{i+1}"] \\
& Z_{i+1}
\end{tikzcd}
$$

III

$$
\begin{tikzcd}
X_i \arrow[rr, dashed, "\phi_i"] \arrow[d, "f_i"'] & & X_{i+1} \arrow[d, "f_{i+1}"] \\
Z_i \arrow[dr, "\alpha_i"'] & & Z_{i+1} \arrow[dl, "\beta_{i+1}"] \\
& T_i &
\end{tikzcd}
$$

IV

All $f_i : X_i \to S_i$ are Mori fibre spaces, all p, q are divisorial contractions, and all dash arrows are a composition of flips with respective to certain $K_{X_i} + B_i$.

Hacon-McKernan's Sarkisov decomposition

After BCHM10, Hacon and McKernan considered this problem in another perspective and proved the following general Sarkisov decomposition:

Theorem (HM)

Let (W, B_W) be a $\mathbb{Q}$-factorial klt pair, and let X, X' be two outputs of MMPs. Suppose $f : (X, B) \to S$ and $f' : (X', B') \to S'$ are two log Mori fibre spaces with the induced birational map $\Phi : X \dashrightarrow X'$.
Then Φ is a composition of finitely many Sarkisov links.

Table of Contents

Double Scaling and Adjoint Foliated Structures

We use another method called *Double Scaling* and prove the Sarkisov program for algebraically integrable foliations and threefold foliations.

Theorem (CLW)

Sarkisov program holds for lc algebraically integrable adjoint foliated structures on $\mathbb{Q}$-factorial klt varieties.
More precisely:
Let $\mathfrak{A}_W/U$ be an lc algebraically integrable adjoint foliated structure whose ambient variety W is $\mathbb{Q}$-factorial klt. Let $g : W \dashrightarrow X$, $g' : W \dashrightarrow X'$ be two $K_{\mathfrak{A}_W}$-MMPs$/U$, $\mathfrak{A} := g_*\mathfrak{A}_W$, $\mathfrak{A}' := g'_*\mathfrak{A}_W$, and let $f : X \to S$, $f' : X' \to S'$ be a $K_{\mathfrak{A}}$-Mori fiber space$/U$ and a $K_{\mathfrak{A}'}$-Mori fiber space$/U$ respectively.
Let Φ be the induced birational map, then Φ is a composition of finitely many Sarkisov links.

The idea of Double Scaling was given in an unpublished note of Hacon, which was later reformulated to obtain the Sarkisov program generalized pairs by Liu.

Table of Contents

Sarkisov degree

Suppose $f : X \to S$ and $g : Y \to T$ are two birational Mori fibre spaces of terminal threefolds, and $\Phi : X \dashrightarrow Y$ is a birational map.
Take an ample divisor A_T on T such that $H_Y \sim -\mu_Y K_Y + g^* A_T$ is a general ample divisor on Y for some $\mu_Y > 0$.
Note that $(Y, \frac{1}{\mu_Y} H_Y)$ is canonical and $K_Y + \frac{1}{\mu_Y} H_Y$ is nef (and trivial over T).
Let H_X be the birational transform of H_Y on X.

Definition (Sarkisov degree)

1. Let $\mu_X = \max\{c \in \mathbb{R} : K_X + \frac{1}{c} H_X \text{ is nef over } S\}$;
2. Let $\lambda_X = \min\{c \in \mathbb{R} : (X, \frac{1}{c} H_X) \text{ is canonical}\}$.

- μ_X compares positivity with respective to H_Y.
- λ_X compares singularity with respective to H_Y.

Theorem (Noether-Fano-Iskovskikh Criterion)

1. *$\mu_X \geqslant \mu_Y$;*
2. *If $\mu_X \geqslant \lambda_X$ (thus $(X, \frac{1}{\mu_X} H_X)$ is canonical) and $(K_X + \frac{1}{\mu_X} H_X)$ is nef, then Φ is an isomorphism of Mori fibre spaces.*

Construction of Sarkisov links I, II

We construct Sarkisov links by induction.
Suppose we have Mori fibre space $f_i : X_i \to S_i$ and Sarkisov degree μ_i, λ_i corresponding to $X_i \dashrightarrow Y$.
Suppose $\lambda_i > \mu_i$, then $(X_i, \frac{1}{\mu_i} H_i)$ has bad singularity. Take a $(K_{X_i} + \frac{1}{\lambda_i} H_i)$-crepant blow-up $V_i \to X_i$ and run a $(K_{V_i} + \frac{1}{\lambda_i} H_i)$-MMP over S_i.
This gives a Sarkisov link of type I or II.

$$\begin{tikzcd}
V_i \arrow[r, dotted, "\phi_i"] \arrow[d, "p_i"'] & X_{i+1} \arrow[d, "f_{i+1}"] \\
X_i \arrow[d, "f_i"'] & S_{i+1} \arrow[dl, "\beta_{i+1}"] \\
S_i &
\end{tikzcd}
\qquad
\begin{tikzcd}
V_i \arrow[r, dotted, "\phi_i"] \arrow[d, "p_i"'] & V_{i+1} \arrow[d, "q_{i+1}"] \\
X_i \arrow[d, "f_i"'] & X_{i+1} \arrow[d, "f_{i+1}"] \\
S_i \arrow[r, "\cong"] & S_{i+1}
\end{tikzcd}$$

I **II**

Construction of Sarkisov links III, IV

Suppose $\lambda_i \leqslant \mu_i$, then $(X_i, \frac{1}{\mu_i}H_i)$ is canonical. Furthermore suppose $K_{X_i} + \frac{1}{\mu_i}H_i$ is not nef. Take a contraction $X_i \to T_i$ with $\rho(X/T) = 2$ and run a $(K_{X_i} + \frac{1}{\mu_i}H_i)$-MMP over T_i.
This gives a Sarkisov link of type III or IV.

$$\begin{tikzcd}
X_i \arrow[r, dashed, "\phi_i"] \arrow[d, "f_i"'] & V_{i+1} \arrow[d, "q_{i+1}"] \\
S_i \arrow[dr, "\alpha_i"'] & X_{i+1} \arrow[d, "f_{i+1}"] \\
& T_i = S_{i+1}
\end{tikzcd}
\qquad
\begin{tikzcd}
X_i \arrow[rr, dashed, "\phi_i"] \arrow[d, "f_i"'] & & X_{i+1} \arrow[d, "f_{i+1}"] \\
S_i \arrow[dr, "\alpha_i"'] & & S_{i+1} \arrow[dl, "\beta_{i+1}"] \\
& T_i &
\end{tikzcd}$$

III **IV**

Both MMPs are 2-ray game. $\rho(X/T) = \rho(V/S) = 2$

Example

Let $\Phi : X = \mathbb{P}^2 \dashrightarrow Y = \mathbb{P}^2$ be a birational map induced by $x \mapsto x, y \mapsto y + x^2$.
Let $H_Y \in |\mathcal{O}_Y(1)|$.
Then we have decomposition:

$$\begin{tikzcd}
 & & & W_2 \arrow[dl] \arrow[dr] & & W_2' \arrow[dl] \arrow[dr] & & & \\
 & & X_1 = \mathbb{F}_1 \arrow[rr, dotted, "\psi_1"] \arrow[d] & & X_2 = \mathbb{F}_2 \arrow[rr, dotted, "\psi_2"] \arrow[d] & & X_3 = \mathbb{F}_1 \arrow[d] \arrow[drr, "\psi_3"] & & \\
X = X_0 = \mathbb{P}^2 \arrow[urr, dotted, "\psi_0"] \arrow[d] & & \mathbb{P}^1 \arrow[rr, equal] \arrow[dll] & & \mathbb{P}^1 \arrow[rr, equal] & & \mathbb{P}^1 \arrow[drr] & & X_4 = Y = \mathbb{P}^2 \arrow[d] \\
\mathrm{pt} & & & & & & & & \mathrm{pt}
\end{tikzcd}$$

Table of Contents

Motivation

After [BCHM10], we have an understanding about MMP and models (called *geography of models*).
We take X, X' as two outputs of $(K_W + B_W)$-MMPs.

- In original Sarkisov program, we have $K_Y + \frac{1}{\mu_Y} H_Y \sim g^* A_T$, therefore we need the same on X.
- For each $X_i \to S_i$ in decomposition, we should expect the same structure, i.e., $K_{X_i} + B_i$ is semi-ample for some B_i and induces $f_i : X \to S_i$.
- All $X, X', X_i, S_i, T_i, V_i$ should be a model of $(W, B_{W,i})$ for some $B_{W,i}$.

Definitions of models

Definition

Let $f : X \dashrightarrow Y$ be a birational map of normal quasi-projective varieties which extracts no divisors, and $p : W \to X$ and $q : W \to Y$ a common resolution. Let D be an $\mathbb{R}$-Cartier divisor on X such that $D_Y = f_* D$ is also $\mathbb{R}$-Cartier. Then f is called D-**non-positive** (respectively D-**negative)** if $E = p^* D - q^* D_Y$ is effective and exceptional over Y (respectively $\operatorname{Supp} p_* E$ contains all f-exceptional divisors).

Definition

Let $\pi : (W, D) \to U$ be a projective morphism of normal quasi-projective varieties, if $K_W + D$ is log canonical and $f : W \dashrightarrow X$ is a birational map extracts no divisors, then

1. X is **weak log canonical model** for $K_W + D$ over U if f is $K_W + D$-non-positive and $K_X + f_* D$ is nef over U;
2. X is **good minimal model** for $K_W + D$ over U if f is $K_W + D$-negative and $K_X + f_* D$ is semi-ample over U; Moreover, $K_X + f_* D$ induces a contraction $X \to S$ and S is the **ample model**.
3. If X is a good minimal model and $K_X + f_* D$ is ample, then X is the ample model of (W, D).

Definitions of polytopes

Then we define the polytopes of divisors:

Definition

Let $\pi : X \to U$ be a projective morphism of normal quasi-projective varieties, and let V be a finite dimensional affine subspace of $\mathrm{WDiv}_{\mathbb{R}}(X)$. Let $A \geqslant 0$ be a fixed ample $\mathbb{R}$-divisor. Define

$$\begin{aligned} V_A &= \{D = A + B : B \in V\} \\ \mathcal{L}_A(V) &= \{D = A + B \in V_A : K_X + D \text{ is log canonical and } B \geqslant 0\} \\ \mathcal{E}_{A,\pi}(V) &= \{D = A + B \in \mathcal{L}_A(V) : K_X + D \text{ is pseudo effective over } U\} \end{aligned}$$

Given a birational contraction $f : \dashrightarrow Y$, define

$$\mathcal{W}_{A,f}(V) = \{D \in \mathcal{E}_A(V) : f \text{ is weak log model of } (X, D) \text{ over } U\}$$

Given a rational contraciton $g : X \dashrightarrow Z$ over U, define

$$\mathcal{A}_{A,g}(V) = \{D \in \mathcal{E}_A(V) : g \text{ is ample model of } (X, D) \text{ over } U\}$$

If U is clear or point, then we drop U and π.

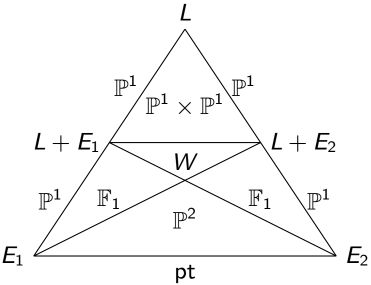
Decomposition of polytopes and morphisms

When the boundary varies in a bounded space, there are finitely many models of W, and there are morphisms between nearby models.

1. There are finitely many birational maps $f_i : W \dashrightarrow X_i$ such that
$$\mathcal{E}_A(V) = \bigcup_i \mathcal{W}_i$$
where $\mathcal{W}_i = \mathcal{W}_{A,f_i}(V)$ is a rational polytope.
2. There are finitely many rational maps $g_j : X \dashrightarrow Z_j$ such that
$$\mathcal{E}_A(V) = \coprod_j \mathcal{A}_j$$
$\{\mathcal{A}_j = \mathcal{A}_{A,g_j}\}$ is a partition of $\mathcal{E}_A(V)$. Let $\mathcal{C}_j$ be the closure of $\mathcal{A}_j$ in $\mathcal{L}_A(V)$;
3. If i, j are two indices such that $\mathcal{A}_j \cap \mathcal{C}_i \neq \emptyset$ then there is a contraction $f_{ij} : X_i \to X_j$ such that $f_j = f_{ij} \circ f_i$;

Example

Let $W \to \mathbb{P}^2$ be the blow-up of two points. Let L be the strict transform of the line connecting two points, and let E_1, E_2 be the exceptional divisors.
Take $V = \mathrm{span}(L, E_1, E_2)$ and $A = -K_W$. Therefore
$K_W + A + aE_1 + bE_2 + cL \sim aE_1 + bE_2 + cL$.
Take a 2-dimensional slice of the space.



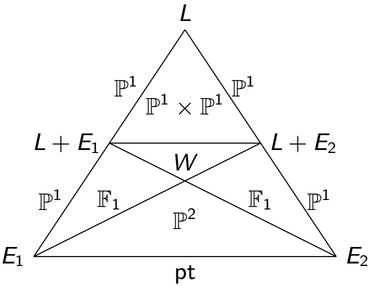
Sarkisov decomposition via polytopes

Suppose $f : X \to S, f' : X' \to S'$ are two outputs of two MMPs on (W, B_W). Hacon-McKernan's method to find the Sarkisov decomposition:

- Take an ample $\mathbb{Q}$-divisor A on S such that $L \sim_{\mathbb{Q}} -(K_X + B) + f^*A$ is a general ample $\mathbb{Q}$-divisor. Similarly, take an ample $\mathbb{Q}$-divisor A' on S' such that $R \sim_{\mathbb{Q}} -(K_{X'} + B') + f'^*A'$ is a general ample $\mathbb{Q}$-divisor.
- Replace W with a common resolution. Moreover, suppose $(W, B_W + L_W + R_W)$ is terminal.
- Then $(X, B + L)$ and $(X', B' + R)$ are two good minimal models of W (for $(W, B_W + L_W)$ and $(W, B_W + R_W)$). And S, S' are ample models respectively.
- Carefully take a 2-dimensional affine space V. The dim $= 2$ implies $\rho = 2$. Some vertexs correspond to a Sarkisov link.
- All morphisms in a Sarkisov link are morphisms between certain models.
- Then Find a path connecting two models and vertexs.

Strictly speaking, this is not a *program*. We just find many models of W and connect them.

Example revisited



For example, at $[L + E_1]$ there is a Sarkisov link $\mathbb{P}^1 \times \mathbb{P}^1 \leftarrow W \rightarrow \mathbb{F}_1$.

$$\begin{tikzcd}
 & W \arrow[dl] \arrow[dr] & & & & W \arrow[dl] \arrow[dr] & & & \\
\mathbb{F}_1 \arrow[d] & & \mathbb{F}_0 \arrow[d] \arrow[rr, equal] & & \mathbb{F}_0 \arrow[d] & & \mathbb{F}_1 \arrow[d] \arrow[dr] & & \mathbb{F}_1 \arrow[d] \arrow[dl] \\
\mathbb{P}^1 & & \mathbb{P}^1 \arrow[dr] & & \mathbb{P}^1 \arrow[dl] & & \mathbb{P}^1 \arrow[dr] & \mathbb{P}^2 \arrow[d] & \mathbb{P}^1 \arrow[dl] \\
 & & & \mathrm{pt} & & & & \mathrm{pt} &
\end{tikzcd}$$

Table of Contents

Motivation of double scaling

Double Scaling is a mixed method.
Let X, X' be two outputs of $(K_W + B_W)$-MMPs.

- We construct two divisors L, R.
- We have a *roof* (W, B_W) of two Mori fibre spaces $f : (X, B) \to S, f' : (X', B') \to S'$.
- Each X_i is a weak log canonical model of $(W, B_W + l_i L_W + r_i R_W)$.
- Each Sarkisov link $X_i \dashrightarrow X_{i+1}$ is constructed by 2-ray game.

Advantages:

- It is a *program*. It is controlled by W, so it terminates.
- We have a clear space $\mathrm{span}(L_W, R_W)$. Do not need details about geography of ample models.

Sketch of Sarkisov decomposition

Replace W with a common resolution of X, X' such that (W, B_W) is terminal. Denote $W(a,b) := (W, B_W + aL_W + bR_W)$. Then X (resp. X') is a good minimal model of $W(1,0)$ (resp. $W(0,1)$).
The trivial case is that $X \cong X'$.
Otherwise, we shift (l, r) in $W(l, r)$ from $(l_0, r_0) = (1, 0)$ to $(l_N, r_N) = (0, 1)$ and construt $h_i : W \dashrightarrow X_i$ and $f : X_i \to S_i$ inductively, with

$$1 = l_0 \geq l_1 \geq \cdots \geq l_N = 0,$$

$$0 = r_0 \leq r_1 \leq \cdots \leq r_N = 1.$$

- $L_i := (h_i)_* L_W, R_i := (h_i)_* R_W, B_i := (h_i)_* B_W$,
  $K(a,b) := K_W + B_W + aL_W + bR_W$ and $K_i(a,b) := (h_i)_* K(a,b)$.
- h_i is $K_W(l_i, r_i)$-non-positive and $(X_i, B_i + l_i L_i + r_i R_i)$ is $\mathbb{Q}$-factorial klt.
- $K_i(l_i, r_i)$ is nef and $K_i(l_i, r_i) \sim_{\mathbb{R}, S_i} 0$.

Table of Contents

Example: common resolution

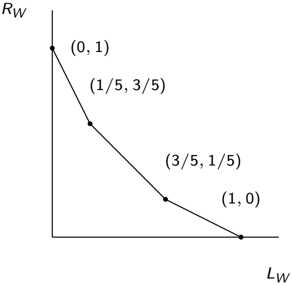
Let $\Phi : X = \mathbb{P}^2 \dashrightarrow X' = \mathbb{P}^2$ be a birational map induced by $x \mapsto x, y \mapsto y + x^2$. Let $B = X \setminus \mathbb{A}^2$ and $B' = X' \setminus \mathbb{A}^2$.
There is a common resolution $\sigma : W \to X$ and $\sigma' : W \to X'$, which are both compositions of three blow-ups at indeterminacy points.

$$
\begin{tikzcd}
 & & & & W_3 = W = W_3' \arrow[dll] \arrow[drr] \arrow[ddd] & & & & \\
 & & W_2 \arrow[dl] \arrow[dr] & & & & W_2' \arrow[dl] \arrow[dr] & & \\
 & W_1 \arrow[dl] & & Z_0 \arrow[dlll] \arrow[dr] & & Z_1 \arrow[dl] \arrow[drrr] & & W_1' \arrow[dr] & \\
X & & & & X_1 & & & & X'
\end{tikzcd}
$$

Let $B_W = \frac{1}{2}(B + E_1 + E_3)$ and consider pairs $(X, \frac{1}{2}B)$ and $(X', \frac{1}{2}B')$. Then we have $L = L_0 \sim_{\mathbb{Q}} \frac{5}{2}B$ and $R \sim_{\mathbb{Q}} \frac{5}{2}B'$.

Example: Sarkisov Program

1. $k_0 = 2$ and $s_0 = \frac{1}{5}$. X_1 is a weak log canonical model of $(W, B_W + \frac{3}{5}L_W + \frac{1}{5}R_W)$;
2. $k_1 = 1$ and $s_1 = \frac{2}{5}$. $X_2 = X'$ is a weak log canonical model of $(W, B_W + \frac{1}{5}L_W + \frac{3}{5}R_W)$.



$$
\begin{tikzcd}
& Z_0 \arrow[dl] \arrow[dr] & & Z_1 \arrow[dl] \arrow[dr] & \\
X \arrow[rr, dashed, "\psi_0"] \arrow[d] & & X_1 \arrow[rr, dashed, "\psi_1"] \arrow[d] & & X' \arrow[d] \\
\mathrm{pt} & & \mathrm{pt} & & \mathrm{pt}
\end{tikzcd}
$$

Table of Contents

Sarkisov Degree in Double Scaling

We decide next node (l_{i+1}, r_{i+1}) from (l_i, r_i) as follows:

Definition

1. C_i is a general f_i-vertical curve.
2. k_i is the unique positive real number such that $R_i \equiv_{Z_i} k_i L_i$.
3. Γ_i is the set of all real numbers t satisfying the following.
   - 3.1 $0 \leq t \leq \frac{l_i}{k_i}$.
   - 3.2 For any divisor $E \subset W$, we have $a(E, W(l_i - k_i t, r_i + t)) \leq a(E, X_i(l_i - k_i t, r_i + t))$. (Singularity condition)
   - 3.3 $K_i(l_i - k_i t, r_i + t)$ is nef. (positivity condition)
4. $s_i := \sup\{t : t \in \Gamma_i\}$.
5. $l_{i+1} := l_i - s_i k_i$ and $r_{i+1} := r_i + s_i$.

Note that

- 1,2,5 imply that $K_i(l_{i+1}, r_{i+1}) \sim_{\mathbb{R}, S_i} 0$.
- 3.2 implies h_i is $K_W(l_{i+1}, r_{i+1})$-non-positive and $(X_i, B_i + l_{i+1} L_i + r_{i+1} R_i)$ is klt.
- 3.3 implies $K_i(l_{i+1}, r_{i+1})$ is nef.

Construction of Sarkisov links

Double scaling Sarkisov program proceeds as follows:

- If $s_i < \frac{l_i}{k_i}$, there are two cases corresponding (3.2) and (3.3) respectively.
  - Assume there is a divisor $E \subset W$ such that for any $0 < \epsilon \ll 1$, we have $a(E, W(l_{i+1} - k_i\epsilon, r_{i+1} + \epsilon)) > a(E, X_i(l_{i+1} - k_i\epsilon, r_{i+1} + \epsilon))$ and $K_i(l_{i+1} - \mu_i\epsilon, r_{i+1} + \epsilon)$ is nef.
    Let $V_i \to X_i$ be an extraction of E and run a MMP on V_i over S_i.
    This gives a Sarkisov link of type i or ii.
    (same as $\lambda_i > \mu_i$ case)
  - Assume that for all $0 < \epsilon \ll 1$, we have $K_i(l_{i+1} - k_i\epsilon, r_{i+1} + \epsilon)$ not nef.
    Take a contraction $X_i \to T_i$ with $\rho(X/T) = 2$ and run a MMP on X_i over T_i.
    This gives a Sarkisov link of type or iii or iv.
    (same as $\lambda_i \leq \mu_i$ and not nef case)
- If $s_i = \frac{l_i}{k_i}$, then we stop the induction and let $N = i + 1$ and $X_N := X_i, S_N := S_i$ and $f_N := f_i : X_N \to Z_N$.
  In this case, $l_N = 0, r_N = 1$ and $(X_N \to S_N) = (X' \to S')$.
  (same as $\lambda_i \leq \mu_i$ and nef case)

Termination of double scaling

The double scaling Sarkisov Program terminates in finite steps.
Otherwise,

- Each $h_i : W \dashrightarrow X_i$ is a weak log canonical model of $(W, B_W + l_i L_W + r_i + R_W)$.
- By finiteness of weak log canonical models, there are $i < j$ such that $X_i \cong X_j$.
- But each Sarkisov link is a 2-ray game MMP, therefore the singularity changes.
- Contraction.

Table of Contents

Compare with other methods

Compare with the original method:

- we compare singularities between X_i and W, not X_i and X';
- all extracted divisors E_i lie on W, which are finite;
- each X_i is a weak log canonical model of certain $W(l_i, r_i)$, which are also finite;

Compare with Hacon-McKernan's method:

- both need a terminal pair $(W, B_W + L_W + R_W)$;
- construct Sarkisov links by 2-ray game, not combining morphisms between ample models;
- only need finiteness of weak log canonical models, do not need geography of ample models.
- do not need to carefully choose the space of boundarie on W.

Table of Contents

Definition of Foliations

Definition (Foliations)

Let X be a normal variety. A *foliation* on X is a coherent sheaf $\mathcal{F} \subset T_X$ such that

1. $\mathcal{F}$ is saturated in $\mathcal{T}_X$, i.e. $\mathcal{T}_X/\mathcal{F}$ is torsion free, and
2. $\mathcal{F}$ is closed under the Lie bracket.

Then

- The *rank* of a foliation $\mathcal{F}$ on a variety X is the rank of $\mathcal{F}$ as a sheaf and is denoted by $\operatorname{rk}\mathcal{F}$. The *co-rank* of $\mathcal{F}$ is $\dim X - \operatorname{rk}\mathcal{F}$.
- The *canonical divisor* of $\mathcal{F}$ is a divisor $K_{\mathcal{F}}$ such that $\mathcal{O}_X(-K_{\mathcal{F}}) \cong \det(\mathcal{F})$. If $\mathcal{F} = 0$, then we say that $\mathcal{F}$ is a *foliation by points*.
- A subvariety $S \subset X$ is called $\mathcal{F}$*-invariant* if for any open subset $U \subset X$ and any section $\partial \in H^0(U, \mathcal{F})$, we have $\partial(\mathcal{I}_{S\cap U}) \subset \mathcal{I}_{S\cap U}$, where $\mathcal{I}_{S\cap U}$ denotes the ideal sheaf of $S \cap U$ in U.
- For any prime divisor P on X, we define $\epsilon_{\mathcal{F}}(P) := 1$ if P is not $\mathcal{F}$-invariant and $\epsilon_{\mathcal{F}}(P) := 0$ if P is $\mathcal{F}$-invariant. For any prime divisor E over X, we define $\epsilon_{\mathcal{F}}(E) := \epsilon_{\mathcal{F}_Y}(E)$ where $h : Y \dashrightarrow X$ is a birational map such that E is on Y and $\mathcal{F}_Y := h^{-1}\mathcal{F}$.

Toy example: Let $f : X \to B$ be a fibration and $\mathcal{T}_{\mathcal{F}} := \ker \mathrm{d}f = \mathcal{T}_{X/B}$.

Definition

Let $\mathfrak{A} = (X, \mathcal{F}, B)$. For any projective birational morphism $h : X' \to X$, we define

$$h^*\mathfrak{A} := (X', \mathcal{F}', B')$$

where $\mathcal{F}' := h^{-1}\mathcal{F}$ and B' is the unique $\mathbb{R}$-divisor such that $K_{\mathcal{F}'} + B' = h^*(K_{\mathcal{F}} + B)$.
For any prime divisor E on X', we denote by

$$a(E, \mathfrak{A}) := -\operatorname{mult}_E B'$$

the *discrepancy* of E with respect to $\mathfrak{A}$.
We say that $\mathfrak{A}$ is *lc* (resp. *klt*) if $a(E, \mathfrak{A}) \geq \epsilon_{\mathcal{F}}(E)$ (resp. $> \epsilon_{\mathcal{F}}(E)$). We say that $\mathfrak{A}$ is *terminal* if $a(E, \mathfrak{A}) > 0$ for any prime divisor E that is exceptional$/X$.

Lemma

Let $\mathfrak{A}/U = (X, \mathcal{F}, B)$ be an algebraically integrable foliation. If $\mathcal{F} \neq T_X$, then $\mathfrak{A}$ is not klt.

Challenges

Our goal is to establish Sarkisov Program for lc algebraically integrable foliations.

- The failure of Bertini-type theorems.
- We have to deal with lc foliations with klt ambient varieties.
- We do not have a good understanding on the geometry of ample models of foliations. So the method of HM fails.
- The double scaling fails because terminalizations usually do not exist for lc foliations.

To overcome these difficulties.

- We can use a simple trick of incorporating the auxiliary ample divisor into the nef part.
- we shall use the theory of *adjoint foliated structures* instead. In short, we consider $(tK_{\mathcal{F}} + (1-t)K_X)$-MMP.

Adjoint Foliated Structures

Definition

An *adjoint foliated structure* $\mathfrak{A}/U := (X, \mathcal{F}, B, \mathbf{M}, t)/U$ is the datum of a normal quasi-projective variety X, a projective morphism $X \to U$, a foliation $\mathcal{F}$ on X, an $\mathbb{R}$-divisor $B \geq 0$ on X, a **b**-divisor $\mathbf{M}$ nef$/U$, and a real number $t \in [0, 1]$ such that

$$K_{\mathfrak{A}} := K_{(X,\mathcal{F},B,\mathbf{M},t)} := tK_{\mathcal{F}} + (1-t)K_X + B + \mathbf{M}_X$$

is $\mathbb{R}$-Cartier. $K_{\mathfrak{A}}$ is called as the *canonical* $\mathbb{R}$*-divisor* of $\mathfrak{A}$. $X, \mathcal{F}, B, \mathbf{M}, t$ are called the *ambient variety*, *foliation part*, *boundary part*, *nef part* (or *moduli part*), and *parameter* of $\mathfrak{A}$ respectively.

Definition

For any projective birational morphism $h : X' \to X$, we define

$$h^*\mathfrak{A} := (X', \mathcal{F}', B', \mathbf{M}, t)$$

where $\mathcal{F}' := h^{-1}\mathcal{F}$ and B' is the unique $\mathbb{R}$-divisor such that $K_{h^*\mathfrak{A}} = h^*K_{\mathfrak{A}}$. For any prime divisor E on X', we denote by

$$a(E, \mathfrak{A}) := -\operatorname{mult}_E B'$$

the *discrepancy* of E with respect to $\mathfrak{A}$. The *total minimal log discrepancy* of $\mathfrak{A}$ is

$$\operatorname{tmld}(\mathfrak{A}) := \inf\{a(E, \mathfrak{A}) + t\epsilon_{\mathcal{F}}(E) + (1-t) \mid E \text{ is over } X\}$$

We say that $\mathfrak{A}$ is *lc* (resp. *klt*) if $\operatorname{tmld}(\mathfrak{A}) \geq 0$ (resp. > 0). We say that $\mathfrak{A}$ is *terminal* if $a(E, \mathfrak{A}) > 0$ for any prime divisor E that is exceptional$/X$.

Table of Contents

Sketch

Recall:
Let $\mathfrak{A}_W = (W, \mathcal{F}_W)$ be an lc algebraically integrable adjoint foliated structure whose ambient variety W is $\mathbb{Q}$-factorial klt. Let $g : W \dashrightarrow X$, $g' : W \dashrightarrow X'$ be two $K_{\mathfrak{A}_W}$-MMPs$/U$, $\mathfrak{A} := g_*\mathfrak{A}_W$, $\mathfrak{A}' := g'_*\mathfrak{A}_W$, and let $f : X \to S$, $f' : X' \to S'$ be a $K_{\mathfrak{A}}$-Mori fiber space$/U$ and a $K_{\mathfrak{A}'}$-Mori fiber space$/U$ respectively.
Any $K_{\mathcal{F}_W}$-MMP can be viewed as a $(tK_{\mathcal{F}_W} + (1-t)K_W + A)$-MMP for some $0 < 1 - t \ll 1$ and an ample $\mathbb{R}$-divisor A, where $(W, \mathcal{F}_W, 0, \overline{A}, t)$ forms a klt adjoint foliated structure.
We need some ingredients to apply Double Scaling on AFS.

- Terminalizations of klt AFS. Therefore we have a good *roof*.
- MMP (with scaling), to construct Sarkisov links.
- Finiteness of weak log canonical models, to show the Sarkisov Program terminates.

MMP

MMP:

Theorem (CHLSSMX25)

Let $\mathfrak{A}/U := (X, \mathcal{F}, B, \mathbf{M}, t)/U$ be a $\mathbb{Q}$-factorial NQC lc algebraically integrable adjoint foliated structure such that X is klt, A an ample$/U$ $\mathbb{R}$-divisor, and let $C \equiv_U D + \mathbf{N}_X$ be an $\mathbb{R}$-divisor on X such that $K_{\mathfrak{A}} + A + C$ is nef$/U$, $(X, \mathcal{F}, B + D, \mathbf{M} + \mathbf{N}, t)$ is lc, and $\mathbf{N}$ is NQC$/U$. Assume that $K_{\mathfrak{A}} + A$ is pseudo-effective$/U$. Then we may run a $(K_{\mathfrak{A}} + A)$-MMP$/U$ with scaling of C and any such MMP terminates with a $\mathbb{Q}$-factorial good minimal model of $(X, \mathcal{F}, B, \mathbf{M} + \overline{A}, t)/U$.

Terminalizations:

Theorem (CLW25)

Let $\mathfrak{A}/U$ be a klt algebraically integrable adjoint foliated structure with ambient variety X. Then there exists a projective birational morphism $h : X' \to X$ such that $\mathfrak{A}' := h^*\mathfrak{A}$ is $\mathbb{Q}$-factorial terminal. The morphism $h : \mathfrak{A}' \to \mathfrak{A}$ is called a $\mathbb{Q}$-factorial terminalization of $\mathfrak{A}$. We also say that $\mathfrak{A}'$ is $\mathbb{Q}$-factorial terminalization of $\mathfrak{A}$.

Polytopes of AFS

Definition

Let $\pi\colon X \to U$ be a projective morphism between normal quasi-projective varieties. A *tuple* of nef$/U$ **b**-Cartier **b**-divisors on X is of the form $\mathfrak{M} := (\mathbf{M}_1, \dots, \mathbf{M}_n)$ for some positive integer n where each $\mathbf{M}_i$ is a nef$/U$ **b**-Cartier **b**-divisor. We define $\dim \mathfrak{M} := n$. We define

$$\mathrm{Span}_{\mathbb{R}}(\mathfrak{M}) := \left\{ \sum_{i=1}^{n} a_i \mathbf{M}_i \,\middle|\, a_i \in \mathbb{R} \right\}, \mathrm{Span}_{\mathbb{R}_{\geq 0}}(\mathfrak{M}) := \left\{ \sum_{i=1}^{n} a_i \mathbf{M}_i \,\middle|\, a_i \in \mathbb{R}_{\geq 0} \right\}.$$

For any nef$/U$ **b**-Cartier **b**-divisor $\mathfrak{N}$, we define $(\mathfrak{M}, \mathfrak{N}) := (\mathbf{M}_1, \dots, \mathbf{M}_n, \mathfrak{N})$. Given a tuple $\mathfrak{M}$ of nef$/U$ **b**-Cartier **b**-divisors on X, let $V \subset \mathrm{Weil}_{\mathbb{R}}(X) \times \mathrm{Span}_{\mathbb{R}}(\mathfrak{M})$ be a finite dimensional affine subspace, $\mathcal{F}$ a foliation on X, and $t \in [0,1]$ a real number. We define:

$$V_{(X,\mathcal{F},t)} := \{\mathfrak{A} \mid \mathfrak{A} = (X, \mathcal{F}, B, \mathbf{M}, t), (B, \mathbf{M}) \in V\}$$

$$\mathcal{L}(V_{(X,\mathcal{F},t)}) := \{\mathfrak{A} \mid \mathfrak{A} = (X, \mathcal{F}, B, \mathbf{M}, t) \in V_{(X,\mathcal{F},t)}, \mathfrak{A} \text{ is lc}, \mathbf{M} \in \mathrm{Span}_{\mathbb{R}_{\geq 0}}(\mathfrak{M})\}.$$

$$\mathcal{E}_{\pi}(V_{(X,\mathcal{F},t)}) := \{\mathfrak{A} \mid \mathfrak{A} \in \mathcal{L}(V_{(X,\mathcal{F},t)}), K_{\mathfrak{A}} \text{ is pseudo-effective}/U\}.$$

Finiteness of weak log canonical models

Theorem (CLW25)

Assume that

- $\pi\colon X \to U$ *is a projective morphism between normal quasi-projective varieties,*
- $\mathfrak{M}$ *is a tuple of nef/*U $\mathbf{b}$*-Cartier* $\mathbf{b}$*-divisors,*
- V *is a finite dimensional rational affine subspace of* $\mathrm{Weil}_{\mathbb{R}}(X) \times \mathrm{Span}_{\mathbb{R}}(\mathfrak{M})$,
- $\mathcal{F}$ *is an algebraically integrable foliation on* X,
- $t \in [0,1]$ *is a rational number.*
- $\mathcal{C} \subset \mathcal{L}(V_{(X,\mathcal{F},t)})$ *is a rational polytope, such that for any* $\mathfrak{A} := (X, \mathcal{F}, B, \mathbf{M}, t) \in \mathcal{C}$, $\mathfrak{A}$ *is klt and* $B + \mathbf{M}_X$ *is big/*U.

Then there are finitely many birational maps/U $\psi_j : X \dashrightarrow Z_j$, $1 \leq j \leq l$ *satisfying the following. For any* $\mathfrak{A} \in \mathcal{C}$ *and any weak lc model* $\mathfrak{A}_Z/U$ *of* $\mathfrak{A}/U$ *with induced birational map* $\psi\colon X \dashrightarrow Z$*, there exists an index* $1 \leq j \leq l$ *such that* $\psi_j \circ \psi^{-1}\colon Z \dashrightarrow Z_j$ *is an isomorphism.*

Proof of Sarkisov Program for AFS

Proof of Main Theorem.

Since g, g' are also $((1-s)K_{\mathfrak{A}_W} + sK_W + H)$-MMPs$/U$ for some $0 < s \ll 1$ and ample $\mathbb{R}$-divisor H. After possibly replacing $\mathfrak{A}_W$, we may assume that $\mathbf{M}$ is NQC$/U$, $t \in [0,1)$, and $\mathfrak{A}$ is klt.
By taking a terminalization, we may assume that $\mathfrak{A}_W$ is $\mathbb{Q}$-factorial terminal, the nef part of $\mathfrak{A}_W$ descends to W, and g, g' are $K_{\mathfrak{A}_W}$-negative morphisms.
Then we can Run Double Scaling Sarkisov Program as in Section 2, replacing W by $\mathfrak{A}_W$.

$\square$

For threefold foliations:

- For rank 2 foliations, R. Mascharak has established the Sarkisov program for $\mathbb{Q}$-factorial foliated F-dlt threefolds. (There is a Bertini-type theorem in this case.)
- For rank 1 foliations on potentially klt threefolds, since $K_{\mathcal{F}}$ is not pseudo-effective, it actually is algebraically integrable ([CP19,ACSS21]).

Thank you