SARKISOV PROGRAM FOR ALGEBRAICALLY INTEGRABLE AND THREEFOLD FOLIATIONS

YIFEI CHEN, JIHAO LIU, AND YANZE WANG

Abstract. By applying the theory of the minimal model program for adjoint foliated structures, we establish the Sarkisov program for algebraically integrable foliations on klt varieties: any two Mori fiber spaces of such structure are connected by a sequence of Sarkisov links. Combining with a result of R. Mascharak, we establish the Sarkisov program for foliations in dimension at most 3 with mild singularities. Log version and adjoint foliated version of the aforementioned Sarkisov programs are also established.

Contents

1. Introduction

We work over the field of complex numbers $\mathbb{C}$.

Sarkisov program of varieties. Let X be an uniruled smooth projective variety. The minimal model program (MMP) predicts that, after running a K_X-MMP, that is, a sequence of K_X-divisorial contractions and flips $X \dashrightarrow X'$, one should obtain a Mori fiber space $X' \to Z$, i.e., a fibration such that $\dim X' > \dim Z$, $-K_{X'}$ is ample$/Z$, and $\rho(X'/Z) = 1$. The existence of such a fibration follows from running a K_X-MMP with scaling of an ample divisor (cf. [BCHM10, Corollary 1.4.2]), though the resulting Mori fiber space is generally not unique. For instance, the Hirzebruch surface $X := \mathbb{F}_1$ admits a natural $\mathbb{P}^1$-fibration structure, which is a K_X-Mori fiber space, yet there also exists a K_X-MMP step $X \to \mathbb{P}^2$, where $\mathbb{P}^2 \to \{\mathrm{pt}\}$ is another Mori fiber space.

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This non-uniqueness naturally raises the question of how different Mori fiber spaces of X are related. The study of such connections, now known as the *Sarkisov program*, was initiated by Sarkisov [Sar80, Sar82] in his work on conic bundles over surfaces. The theory was further developed by Corti [Cor95], Iskovskikh [Isk96], and Bruno-Matsuki [BM97], among others, in the context of the MMP for klt pairs. Roughly speaking, the Sarkisov program aims to decompose any birational map $X_1 \dashrightarrow X_2$ between Mori fiber spaces into a sequence of elementary birational transformations called *Sarkisov links* (see Definition 3.3 for a precise description). This decomposition plays an important role in the study of the birational automorphism groups of Mori fiber spaces associated with Fano varieties and the study of higher-dimensional Cremona groups (cf. [Isk91, IKT93, Isk96, LZ20, BLZ21]).

Following the proof of the existence of the minimal model program for klt varieties [BCHM10], Hacon and McKernan [HM13, Theorem 1.1] showed that any two Mori fiber spaces $X_1 \to Z_1$ and $X_2 \to Z_2$ that are outputs of a K_X-MMP are connected by a sequence of Sarkisov links. An alternative proof was given in an unpublished note of Hacon [Hac12], which was later reformulated to obtain the Sarkisov program for generalized pairs by the second author [Liu21] (see [CW25] by the first and the third author for a survey). Variations of the Sarkisov program have since been developed in several directions, such as the group-equivariant Sarkisov program [Flo20] and the Sarkisov program in positive and mixed characteristics for low-dimensional varieties [BFSZ24]. For further related work, we refer the reader to [Kal13, Miy19, FP22, He24].

Sarkisov program for foliations. In recent years, there has been substantial progress in establishing the foundations of the minimal model program (MMP) for algebraically integrable foliations and threefold foliations. For an lc foliation $\mathcal{F}$ of rank r on a klt variety X of dimension d, the existence of a $K_{\mathcal{F}}$-MMP (i.e. the cone theorem, contraction theorem, and existence of flips) has been established when $d = 3$ [McQ08, Bru15], when $(r, d) = (2, 3)$ and $\mathcal{F}$ is $\mathbb{Q}$-factorial F-dlt [Spi20, CS21, SS22], when $(r, d) = (1, 3)$ and X is $\mathbb{Q}$-factorial [CS25], and when $\mathcal{F}$ is algebraically integrable, regardless of (r, d) [CHLX23, LMX24, CHLMSSX24].

Given these developments, it becomes natural to investigate whether the Sarkisov program admits a counterpart for foliations. Just as the classical Sarkisov program is deeply connected to the study of birational automorphism groups of Fano varieties, its foliated version is expected to play an important role in understanding birational automorphism groups of foliations. A recent work by R. Mascharak [Mas24, Theorem 1.1] has established the Sarkisov program for foliations for the case $(r, d) = (2, 3)$, assuming that $\mathcal{F}$ has mild singularities ($\mathbb{Q}$-factorial foliated F-dlt). While not explicitly stated, the methods employed in the proof extend naturally to the $(r, d) = (1, 2)$ case. However, the $(r, d) = (1, 3)$ case remains largely open due to the failure of Bertini-type theorems, despite some partial results, e.g., [Mas24, Theorem 1.3].

Flops connect minimal models. It is also worth mentioning another case. When the (log) canonical divisor is pseudo-effective, the MMP predicts that the process ends with a minimal model. Such minimal models are in general not unique. Kawamata proved that for terminal pairs, any two minimal models can be connected by a sequence of flops [Kaw08]. Similar results hold for generalized lc pairs [Has20],[Cha24]. Furthermore, analogous statements have been established for some algebraically integrable and threefold foliations [CLW25],[JV23].

The first goal of this paper is to establish the Sarkisov program for rank 1 lc foliations on $\mathbb{Q}$-factorial klt threefolds.

Theorem 1.1. *Sarkisov program holds for lc foliations of rank* 1 *on $\mathbb{Q}$-factorial projective klt threefolds. More precisely, let $\mathcal{F}$ be an lc foliation of rank* 1 *on a $\mathbb{Q}$-factorial projective klt threefold X. For $i \in \{1,2\}$, let $(X,\mathcal{F}) \dashrightarrow (X_i,\mathcal{F}_i)$ be two $K_{\mathcal{F}}$-MMPs and let $f_i : X_i \to Z_i$ be $K_{\mathcal{F}_i}$-Mori fiber spaces. Then the f_1 and f_2 are connected by a finite sequence of Sarkisov links.*

Combining with [Mas24, Theorem 1.1], we obtain the Sarkisov program in dimension ≤ 3 for foliations with mild singularities. Here "mild" stands for "lc foliations on $\mathbb{Q}$-factorial klt varieties" unless $\operatorname{rank}\mathcal{F} = 2$ and $\dim X = 3$, and stands for "$\mathbb{Q}$-factorial F-dlt foliations" ([CS21, Definition 3.1]) if $\operatorname{rank}\mathcal{F} = 2$ and $\dim X = 3$. In either case, these foliations are always outputs of foliated log smooth foliations.

We have the following precise result:

Theorem 1.2. *Sarkisov program holds for foliated log smooth foliations in dimension ≤ 3.*

Theorem 1.1 is actually a very special case of the following result on the Sarkisov program for $\mathbb{Q}$-factorial klt algebraically integrable adjoint foliated structures.

Theorem 1.3. *Sarkisov program holds for lc algebraically integrable adjoint foliated structures (see Definition 2.5) on $\mathbb{Q}$-factorial klt varieties. More precisely:*

Let $\mathfrak{A}_W/U$ be an lc algebraically integrable adjoint foliated structure whose ambient variety W is $\mathbb{Q}$-factorial klt. Let $g : W \dashrightarrow X$, $g' : W \dashrightarrow X'$ be two $K_{\mathfrak{A}_W}$-MMPs$/U$, $\mathfrak{A} := g_*\mathfrak{A}_W$, $\mathfrak{A}' := g'_*\mathfrak{A}_W$, and let $f : X \to Z$, $f' : X' \to Z'$ be a $K_{\mathfrak{A}}$-Mori fiber space$/U$ and a $K_{\mathfrak{A}'}$-Mori fiber space$/U$ respectively. Then the f and f' are connected by a finite sequence of Sarkisov links$/U$.

As a special case of Theorem 1.3, we obtain:

Theorem 1.4. *Sarkisov program holds for lc algebraically integrable foliations on $\mathbb{Q}$-factorial klt varieties.*

We outline the key ideas in the proof of Theorem 1.1. First we take a terminal common resolution $\mathfrak{A}_W$ of two Mori fiber spaces $f : X \to Z, f' : X' \to Z'$, and take two ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisor L, R on X, X' respectively, then X (resp. X') is the ample model of $(\mathfrak{A}_W, \overline{L})$ (resp. $(\mathfrak{A}_W, \overline{R})$). The Sarkisov program constructs $f_i : X_i \to Z_i$ inductively such that X_i is a weak lc model of $(\mathfrak{A}_W, l_i\overline{L} + r_i\overline{R})$, and $X_i \dashrightarrow X_{i+1}$ is a Sarkisov link.

The main challenge in establishing the Sarkisov program for rank 1 foliations on threefolds is the failure of Bertini-type theorems [Mas24]. While recent advances in the minimal model program for foliations suggest that such failures can often be circumvented by working with generalized foliated quadruples [LLM23, CHLX23, LMX24] via the simple trick of incorporating the auxiliary ample divisor into the nef part, this approach is ineffective for the Sarkisov program. There are two major difficulties: First, the foliation in the Sarkisov program cannot be assumed to be klt, so the classical strategy of [HM13] is inapplicable, and we do not have a good understanding on the geometry of ample models of foliations. Second, the alternative approach "Sarkisov program with double scaling" from [Liu21] fails because terminalizations usually do not exist for lc foliations.

To overcome these difficulties, in this paper, we shall use the theory of *adjoint foliated structures* instead. The crucial observation is that any $K_{\mathcal{F}}$-MMP can be viewed as a

$(tK_{\mathcal{F}}+(1-t)K_X+A)$-MMP for some $0<1-t\ll 1$ and an ample $\mathbb{R}$-divisor A, where $(X,\mathcal{F},0,\overline{A},t)$ forms a klt adjoint foliated structure. In this case, we can replace the category of "lc foliations on klt varieties" with the category of "klt adjoint foliated structures". The latter is a more technical and complicated category, but thanks to the recent development on the minimal model program for algebraically integrable adjoint foliated structures [CHLMSSX24, CHLMSSX25], we can apply the arguments as in [Liu21] to deduce the corresponding Sarkisov program (Theorem 1.3). We conclude by noting that all rank 1 foliations appearing in the Sarkisov program for potentially klt varieties have non-pseudo-effective canonical bundles and are consequently algebraically integrable [CP19, ACSS21].

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2. Preliminaries

We will adopt the standard notation and definitions on MMP in [KM98, BCHM10] and use them freely. For adjoint foliated structures, generalized foliated quadruples, foliated triples, foliations, and generalized pairs, we adopt the notation and definitions in [CHLMSSX24, CHLMSSX25] which generally align with [LLM23, CHLX23] (for generalized foliated quadruples), [ACSS21, CS21, CS25] (for foliations and foliated triples), and [BZ16, HL23] (for generalized pairs and $\boldsymbol{b}$-divisors), possibly with minor differences.

2.1. Notation.

Definition 2.1. A *contraction* is a projective morphism of varieties $f\colon X\to Y$ such that $f_*\mathcal{O}_X=\mathcal{O}_Y$. When X,Y are normal, the condition $f_*\mathcal{O}_X=\mathcal{O}_Y$ is equivalent to the fibers of f being connected.

Notation 2.2. Let $h\colon X\dashrightarrow X'$ be a birational map between normal varieties. By abusing notation, we denote by $\mathrm{Exc}(h)$ the reduced divisor supported on the codimension one part of the exceptional locus of h.

Definition 2.3 (NQC)**.** Let $X\to U$ be a projective morphism between normal quasi-projective varieties. Let D be a nef $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X and $\mathbf{M}$ a nef $\boldsymbol{b}$-divisor on X. We say that D is *NQC*$/U$ if $D=\sum d_iD_i$, where each $d_i\geq 0$ and each D_i is a nef$/U$ Cartier divisor. Here NQC stands for "Nef $\mathbb{Q}$-Cartier Combinations", cf. [HL22, Definition 2.15]. We say that $\mathbf{M}$ is *NQC*$/U$ if $\mathbf{M}=\sum\mu_i\mathbf{M}_i$, where each $\mu_i\geq 0$ and each $\mathbf{M}_i$ is a nef$/U$ $\boldsymbol{b}$-Cartier $\boldsymbol{b}$-divisor.

2.2. Adjoint foliated structures.

Definition 2.4 (Foliations, cf. [Bru15, ACSS21, CS21])**.** Let X be a normal variety. A *foliation* on X is a coherent sheaf $\mathcal{F}\subset T_X$ such that

(1) $\mathcal{F}$ is saturated in T_X, i.e. $T_X/\mathcal{F}$ is torsion free, and
(2) $\mathcal{F}$ is closed under the Lie bracket.

The *rank* of a foliation $\mathcal{F}$ on a variety X is the rank of $\mathcal{F}$ as a sheaf and is denoted by $\operatorname{rank}\mathcal{F}$. The *co-rank* of $\mathcal{F}$ is $\dim X-\operatorname{rank}\mathcal{F}$. The *canonical divisor* of $\mathcal{F}$ is a divisor $K_{\mathcal{F}}$ such that $\mathcal{O}_X(-K_{\mathcal{F}})\cong\det(\mathcal{F})$. If $\mathcal{F}=0$, then we say that $\mathcal{F}$ is a *foliation by points*.

Given any dominant map $h : Y \dashrightarrow X$ and a foliation $\mathcal{F}$ on X, we denote by $h^{-1}\mathcal{F}$ the *pullback* of $\mathcal{F}$ on Y as constructed in [Dru21, 3.2] and say that $h^{-1}\mathcal{F}$ is *induced by* $\mathcal{F}$. Given any birational map $g : X \dashrightarrow X'$, we denote by $g_*\mathcal{F} := (g^{-1})^{-1}\mathcal{F}$ the *pushforward* of $\mathcal{F}$ on X' and also say that $g_*\mathcal{F}$ is *induced by* $\mathcal{F}$. We say that $\mathcal{F}$ is an *algebraically integrable foliation* if there exists a dominant map $f : X \dashrightarrow Z$ such that $\mathcal{F} = f^{-1}\mathcal{F}_Z$, where $\mathcal{F}_Z$ is the foliation by points on Z, and we say that $\mathcal{F}$ is *induced by* f.

A subvariety $S \subset X$ is called $\mathcal{F}$*-invariant* if for any open subset $U \subset X$ and any section $\partial \in H^0(U, \mathcal{F})$, we have $\partial(\mathcal{I}_{S\cap U}) \subset \mathcal{I}_{S\cap U}$, where $\mathcal{I}_{S\cap U}$ denotes the ideal sheaf of $S \cap U$ in U. For any prime divisor P on X, we define $\epsilon_{\mathcal{F}}(P) := 1$ if P is not $\mathcal{F}$-invariant and $\epsilon_{\mathcal{F}}(P) := 0$ if P is $\mathcal{F}$-invariant. For any prime divisor E over X, we define $\epsilon_{\mathcal{F}}(E) := \epsilon_{\mathcal{F}_Y}(E)$ where $h : Y \dashrightarrow X$ is a birational map such that E is on Y and $\mathcal{F}_Y := h^{-1}\mathcal{F}$.

If foliation structure $\mathcal{F}$ on X is clear in the context, then given an $\mathbb{R}$-divisor $D = \sum_{i=1}^k a_iD_i$ where each D_i is a prime divisor, we denote by $D^{\text{ninv}} := \sum \epsilon_{\mathcal{F}}(D_i)a_iD_i$ the *non-*$\mathcal{F}$*-invariant part* of D and $D^{\text{inv}} := D - D^{\text{ninv}}$ the $\mathcal{F}$*-invariant part* of D.

Definition 2.5. An *adjoint foliated structure* $\mathfrak{A}/U := (X, \mathcal{F}, B, \mathbf{M}, t)/U$ is the datum of a normal quasi-projective variety X, a projective morphism $X \to U$, a foliation $\mathcal{F}$ on X, an $\mathbb{R}$-divisor $B \geq 0$ on X, a $\boldsymbol{b}$-divisor $\mathbf{M}$ nef$/U$, and a real number $t \in [0, 1]$ such that

$$K_{\mathfrak{A}} := K_{(X,\mathcal{F},B,\mathbf{M},t)} := tK_{\mathcal{F}} + (1-t)K_X + B + \mathbf{M}_X$$

is $\mathbb{R}$-Cartier. $K_{\mathfrak{A}}$ is called as the *canonical* $\mathbb{R}$*-divisor* of $\mathfrak{A}$. $X, \mathcal{F}, B, \mathbf{M}, t$ are called the *ambient variety*, *foliation part*, *boundary part*, *nef part* (or *moduli part*), and *parameter* of $\mathfrak{A}$ respectively. The $\mathbb{R}$-divisor $B + \mathbf{M}_X$ is called the *generalized boundary* of $\mathfrak{A}$. We say that $\mathfrak{A}/U$ is *of general type* if $K_{\mathfrak{A}}$ is big$/U$.

For any $\mathbb{R}$-divisor D on X and $\boldsymbol{b}$-divisor $\mathbf{N}$ on X such that $D + \mathbf{N}_X$ is $\mathbb{R}$-Cartier and $\mathbf{M} + \mathbf{N}$ is nef$/U$, we denote by $(\mathfrak{A}, D, \mathbf{N}) := (X, \mathcal{F}, B + D, \mathbf{M} + \mathbf{N}, t)$. If $D = 0$ then we may drop D, and if $\mathbf{N} = \mathbf{0}$ then we may drop $\mathbf{N}$. For any projective birational morphism $h : X' \to X$, we define

$$h^*\mathfrak{A} := (X', \mathcal{F}', B', \mathbf{M}, t)$$

where $\mathcal{F}' := h^{-1}\mathcal{F}$ and B' is the unique $\mathbb{R}$-divisor such that $K_{h^*\mathfrak{A}} = h^*K_{\mathfrak{A}}$. For any birational map$/U$ $\phi : X \dashrightarrow Y$ which does not extract any divisor and any adjoint foliated structure $\mathfrak{A}/U$ on X, we define $\phi_*\mathfrak{A}/U$ to be the adjoint foliated structure such that $\phi_*\mathfrak{A} := (Y, \phi_*\mathcal{F}, \phi_*B, \mathbf{M}, t)$ and say that $\phi_*\mathfrak{A}$ is the *image* of $\mathfrak{A}$ on Y. For any prime divisor E on X', we denote by

$$a(E, \mathfrak{A}) := -\operatorname{mult}_E B'$$

the *discrepancy* of E with respect to $\mathfrak{A}$. The *total minimal log discrepancy* of $\mathfrak{A}$ is

$$\operatorname{tmld}(\mathfrak{A}) := \inf\{a(E, \mathfrak{A}) + t\epsilon_{\mathcal{F}}(E) + (1-t) \mid E \text{ is over } X\}$$

We say that $\mathfrak{A}$ is *lc* (resp. *klt*) if $\operatorname{tmld}(\mathfrak{A}) \geq 0$ (resp. > 0). We say that $\mathfrak{A}$ is *terminal* if $a(E, \mathfrak{A}) > 0$ for any prime divisor E that is exceptional$/X$. If we allow B to have negative coefficients, then we add the prefix "sub-" for the types of singularities above.

When $\mathbf{M} = \mathbf{0}$, and either $t = 0$ or $\mathcal{F} = T_X$, we call $(X, B)/U$ a *pair*. We say that $(X, \mathcal{F}, B, \mathbf{M}, t)/U$ is *NQC* if $\mathbf{M}$ is NQC$/U$. If $B = 0$, or if $\mathbf{M} = \mathbf{0}$, or if U is not important, then we may drop $B, \mathbf{M}, U$ respectively. If $U = \{pt\}$ then we also drop U and say that $(X, \mathcal{F}, B, \mathbf{M}, t)$ is *projective*. If we allow B to have negative coefficients, then we shall add the prefix "sub-". If B is a $\mathbb{Q}$-divisor, $\mathbf{M}$ is a $\mathbb{Q}$-$\boldsymbol{b}$-divisor, and $t \in \mathbb{Q}$, then we shall add the prefix "$\mathbb{Q}$-".

2.3. **Birational maps in the MMP.**

Definition 2.6. Let $X \to U$ be a projective morphism from a normal quasi-projective variety to a variety. Let D be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X and $\phi : X \dashrightarrow X'$ a birational map$/U$. Then we say that X' is a *birational model* of X. We say that ϕ is D-non-positive (resp. D-negative, D-trivial, D-non-negative, D-positive) if the following conditions hold:

(1) ϕ does not extract any divisor.
(2) $D' := \phi_* D$ is $\mathbb{R}$-Cartier.
(3) There exists a resolution of indeterminacy $p : W \to X$ and $q : W \to X'$, such that

$$p^* D = q^* D' + F$$

where $F \geq 0$ (resp. $F \geq 0$ and $\operatorname{Supp} p_* F = \operatorname{Exc}(\phi)$, $F = 0$, $0 \geq F$, $0 \geq F$ and $\operatorname{Supp} p_* F = \operatorname{Exc}(\phi)$).

Definition 2.7. Let $f : X \to Z$ be a contraction$/U$ between normal quasi-projective varieties, and D an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. f is called a *D-Mori fiber space* if $\dim X > \dim Z$, $\rho(X/Z) = 1$, and D is anti-ample$/Z$. If f is a D-Mori fiber space for some D, then we say that f is a *Mori fiber space*.

Definition 2.8. Let $X \to U$ be a projective morphism from a normal quasi-projective variety to a variety. Let D be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X, $\phi : X \dashrightarrow X'$ a D-non-positive birational map$/U$, and $D' := \phi_* D$.

(1) If D' is nef$/U$, then X' is called a *weak lc model$/U$* of D.
(2) If D' is nef$/U$ and ϕ is D-negative, then X' is called a *minimal model$/U$* of D.
(3) If ϕ is D-negative and D' is semi-ample$/U$, then X' is called a *good minimal model$/U$* of D.
(4) Assume that there exists a contraction$/Z$ $f : X' \to Z$ that is a D'-Mori fiber space, and ϕ is D-negative. Then f is called a *Mori fiber space$/U$* of D.

Definition 2.9. Let $\mathfrak{A}/U$ be an adjoint foliated structure with ambient variety X and let $\phi : X \dashrightarrow X'$ be a birational map$/U$. Let $\mathfrak{A}' := \phi_* \mathfrak{A}$. We say that $\mathfrak{A}'/U$ is *weak lc model* (resp. *minimal model, good minimal model*) of $\mathfrak{A}/U$ if X' is a weak lc model$/U$ (resp. minimal model$/U$, good minimal model$/U$) of $K_{\mathfrak{A}}$. A contraction$/U$ $f : X' \to Z$ is called a *Mori fiber space* of $\mathfrak{A}/U$ if f is a Mori fiber space$/U$ of $K_{\mathfrak{A}}$. We say that f is *$\mathbb{Q}$-factorial* if X' is $\mathbb{Q}$-factorial.

Definition 2.10 (cf. [ACSS21, 3.2 Log canonical foliated pairs], [CHLX23, Definition 6.2.1])**.** Let $\mathfrak{A}/U := (X, \mathcal{F}, B, \mathbf{M}, t)/U$ be an algebraically integrable adjoint foliated structure. We say that $\mathfrak{A}$ is *foliated log smooth* if there exists a contraction $f : X \to Z$ satisfying the following.

(1) X has at most quotient toric singularities.
(2) $\mathcal{F}$ is induced by f.
(3) (X, Σ_X) is toroidal for some reduced divisor Σ_X such that $\operatorname{Supp} B \subset \Sigma_X$. In particular, $(X, \operatorname{Supp} B)$ is toroidal, and X is $\mathbb{Q}$-factorial klt.
(4) There exists a log smooth pair (Z, Σ_Z) such that

$$f : (X, \Sigma_X) \to (Z, \Sigma_Z)$$

is an equidimensional toroidal contraction.
(5) $\mathbf{M}$ descends to X.

We say that $f:(X,\Sigma_X)\to(Z,\Sigma_Z)$ is *associated with* $(X,\mathcal{F},B,\mathbf{M},t)$. It is important to remark that f may not be a contraction$/U$. In particular, $\mathbf{M}$ may not be nef$/Z$.

Note that the definition of foliated log smooth has nothing to do with t. In other words, as long as $(X,\mathcal{F},B,\mathbf{M})$ is foliated log smooth, $(X,\mathcal{F},B,\mathbf{M},t)$ is foliated log smooth.

Definition 2.11 (Foliated log resolutions)**.** Let $\mathfrak{A}/U:=(X,\mathcal{F},B,\mathbf{M},t)/U$ be an algebraically integrable sub-adjoint foliated structure. A *foliated log resolution* of $\mathfrak{A}$ is a birational morphism $h:X'\to X$ such that

$$(X',\mathcal{F}':=h^{-1}\mathcal{F},B':=h^{-1}_*B+\operatorname{Exc}(h),\mathbf{M},t)$$

is foliated log smooth. By [CHLX23, Lemma 6.2.4], foliated log resolution for $\mathfrak{A}$ always exists.

2.4. **Terminalization.**

Lemma 2.12. *Let $\mathfrak{A}/U=(X,\mathcal{F},B,\mathbf{M},t)$ be an algebraically integrable sub-adjoint foliated structure. If $t=1$ and $\mathcal{F}\neq T_X$, then $\mathfrak{A}$ is not sub-klt.*

Proof. There exists an $\mathcal{F}$-invariant divisor D on X such that D is not a component of $\operatorname{Supp}B$. We have $a(D,\mathfrak{A})=0$, so $\mathfrak{A}$ is not sub-klt. $\square$

Lemma 2.13. *Let $\mathfrak{A}/U$ be a sub-klt algebraically integrable sub-adjoint foliated structure with ambient variety X. Then*

$$\mathcal{S}_0:=\{F\mid F\text{ is exceptional}/X, a(F,\mathfrak{A})\leq 0\}$$

is a finite set.

Proof. After possibly replacing $\mathfrak{A}$ with a foliated log resolution, we may assume that $\mathfrak{A}=(X,\mathcal{F},B,\mathbf{M},t)$ is foliated log smooth, and then we may assume that $\mathbf{M}=\mathbf{0}$. We may assume that $\dim X\geq 2$.

If $t=1$, by Lemma 2.12, we have $\mathcal{F}=T_X$, and $\mathfrak{A}$ is a usual klt pair (X,B). Take a log resolution $\phi:(Y,B_Y)\to(X,B)$ with $K_Y+B_Y=\phi^*(K_X+B)$. Furthermore, assume that $B_Y=B_Y^+-B_Y^-$, where $B_Y^+,B_Y^-\geq 0$ has no common component and all irreducible components of B_Y^+ are disjoint. Then (Y,B_Y^+) is terminal and any exceptional divisor over X with non-positive discrepancy is a component of B_Y^+. Therefore the set is finite. See also [BCHM10, Corollary 1.4.3].

Assume $t<1$. Let $f:(X,\Sigma_X)\to(Z,\Sigma_Z)$ be a contraction associated with $(X,\mathcal{F},B,t)$ and let Σ_X^h,Σ_X^v be the horizontal$/Z$ and vertical$/Z$ part respectively. Since $\mathfrak{A}$ is klt, there exists a real number $\epsilon\in(0,1)$ such that

$$(1-\epsilon)(\Sigma_X^h+(1-t)\Sigma_X^v)\geq B.$$

After possibly replacing B, we may assume that $(1-\epsilon)(\Sigma_X^h+(1-t)\Sigma_X^v)=B$. Then we have

$$K_{\mathfrak{A}}=t(K_{\mathcal{F}}+(1-\epsilon)\Sigma_X^h)+(1-t)(K_X+(1-\epsilon)\Sigma_X).$$

Let

$$\Delta:=(1-\epsilon)\Sigma_X^h+(1-\epsilon+t\epsilon)\Sigma_X^v,$$

then (X,Δ) is klt, and by [ACSS21, Proposition 3.6], we have

$$K_X+\Delta\sim_{\mathbb{R},Z}K_{\mathfrak{A}}.$$

Note that the set

$$\mathcal{S}:=\{E\mid E\text{ is exceptional}/X, a(E,X,\Delta)\leq 0\}$$

is a finite set. Now for any prime divisor F that is exceptional$/X$ and $a(F,\mathfrak{A})\leq 0$, there are two possibilities:

Case 1. $V:=\operatorname{center}_X F$ dominates Z. Let $g: X'\to X$ be a birational morphism such that F is on X', $\mathfrak{A}':=g^*\mathfrak{A}$, and $K_{X'}+\Delta':=g^*(K_X+\Delta)$. Then over the generic point of V we have $K_{X'}+\Delta'=K_{\mathfrak{A}'}$ and $K_X+\Delta=K_{\mathfrak{A}}$, hence

$$a(F,\mathfrak{A})=a(F,\mathfrak{A}')=a(F,X',\Delta')=a(F,X,\Delta).$$

Therefore, $F\in\mathcal{S}$.

Case 2. $V:=\operatorname{center}_X F$ does not dominate Z. Then F is $\mathcal{F}$-invariant. Since $(X,\mathcal{F},B)$ is foliated log smooth, $a(F,X,\mathcal{F},(1-\epsilon)\Sigma_X^h)\geq 0$. Since $a(F,\mathfrak{A})\leq 0$,

$$a(F,X,\Delta)\leq a(F,X,(1-\epsilon)\Sigma_X)\leq 0$$

by linearity of discrepancies. Thus $F\in\mathcal{S}$.

So $\mathcal{S}_0\subset\mathcal{S}$ is a finite set and we are done. $\square$

Lemma 2.14. *Let $\mathfrak{A}/U$ be a terminal and klt algebraically integrable sub-adjoint foliated structure with ambient variety X and parameter $t<1$, and $D\geq 0$ an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. Then $(\mathfrak{A},\epsilon D)$ is terminal and klt for any $0<\epsilon\ll 1$.*

Proof. By [CHLMSSX25, Lemma 3.25], there exists $\epsilon_0>0$ such that $(\mathfrak{A},\epsilon_0 D)$ is klt. By Lemma 2.13,

$$\mathcal{S}_0:=\{F\mid F\text{ is exceptional}/X, a(F,(\mathfrak{A},\epsilon_0 D))\leq 0\}$$

is a finite set. For any $F\in\mathcal{S}_0$, there exists $\epsilon_F>0$ such that $a(F,(\mathfrak{A},\epsilon_F D))>0$. Now $(\mathfrak{A},\epsilon D)$ is terminal for any $0<\epsilon<\min_F\{\epsilon_F\}$. $\square$

Definition-Theorem 2.15. Let $\mathfrak{A}/U$ be a klt algebraically integrable sub-adjoint foliated structure with ambient variety X. Then there exists a projective birational morphism $h: X'\to X$ such that $\mathfrak{A}':=h^*\mathfrak{A}$ is $\mathbb{Q}$-factorial terminal. The morphism $h:\mathfrak{A}'\to\mathfrak{A}$ is called a *$\mathbb{Q}$-factorial terminalization* of $\mathfrak{A}$. We also say that $\mathfrak{A}'$ is $\mathbb{Q}$-factorial terminalization of $\mathfrak{A}$.

Proof. By Lemma 2.13, $\mathcal{S}_0:=\{F\mid F\text{ is exceptional}/X, a(F,\mathfrak{A})\leq 0\}$ is a finite set. By [CHLMSSX25, Theorem 2.2.3], there exists a projective birational morphism $h: X'\to X$ such that X' is $\mathbb{Q}$-factorial and the divisors contracted by h are exactly the divisors in $\mathcal{S}_0$. Therefore, h satisfies our requirements. $\square$

Definition 2.16. Let $\mathfrak{A}/U$ and $\mathfrak{A}'/U$ be two algebraically integrable adjoint foliated structures with ambient varieties X and X' respectively associated with birational map $\phi: X\dashrightarrow X'$ such that $\mathfrak{A}'=\phi_*\mathfrak{A}$. If

(1) ϕ does not extract any divisor,
(2) $a(E,\mathfrak{A})\leq a(E,\mathfrak{A}')$ for any prime divisor E over X',

then we write $\mathfrak{A}\geq\mathfrak{A}'$.

Lemma 2.17. *Let $\mathfrak{A}/U$ and $\mathfrak{A}'/U$ be two algebraically integrable adjoint foliated structures with ambient varieties X and X' respectively associated with birational map $\phi: X\dashrightarrow X'$ such that $\mathfrak{A}'=\phi_*\mathfrak{A}$. Let $h:\mathfrak{A}''\to\mathfrak{A}'$ be a $\mathbb{Q}$-factorial terminalization of $\mathfrak{A}'$ and X'' the ambient variety of $\mathfrak{A}''$.*

Assume that $K_{\mathfrak{A}'}$ is nef$/U$, ϕ does not extract any divisor, $\mathfrak{A}$ is terminal, and $a(E,\mathfrak{A})\leq a(E,\mathfrak{A}')$ for any prime divisor E on X. Then $\mathfrak{A}\geq\mathfrak{A}'$, $\mathfrak{A}'$ is klt, and $\psi: X\dashrightarrow X''$ does not extract any divisor.

Proof. Let $p : W \to X$ and $q : W \to X'$ be a resolution of indeterminacy of ϕ and write $p^*K_{\mathfrak{A}} = q^*K_{\mathfrak{A}'} + D$ for some D. Since $a(E, \mathfrak{A}) \leq a(E, \mathfrak{A}')$ for any prime divisor E on X, $p_*D \geq 0$. Since D is anti-nef$/X$, by the negativity lemma, $D \geq 0$. Thus $\mathfrak{A} \geq \mathfrak{A}'$ and $\mathfrak{A}'$ is klt.

Suppose that ψ extracts a prime divisor E, then E is exceptional$/X$ and exceptional$/X'$. Since $\mathfrak{A}$ is terminal,

$$0 < a(E, \mathfrak{A}) \leq a(E, \mathfrak{A}') \leq 0$$

which is not possible. □

Lemma 2.18. *Let $\mathfrak{A}/U$ be a klt algebraically integrable adjoint foliated structure and let $f_i : X_i \to Z_i$, $i \in \{1, 2\}$ be two $\mathbb{Q}$-factorial Mori fiber spaces of $\mathfrak{A}/U$ associated with birational maps $\phi_i : X \dashrightarrow X_i$. Let $\mathfrak{A}_i := (\phi_i)_*\mathfrak{A}$.*

Then there exists a $\mathbb{Q}$-factorial terminal algebraically integrable adjoint foliated structure $\bar{\mathfrak{A}}/U$ with ambient variety $\overline{X}$ and projective birational morphisms $h_i : \overline{X} \to X_i$, $i \in \{1, 2\}$, such that $\mathbf{M}$ descends to $\overline{X}$, and h_i can be decomposed into a sequence of steps of a $K_{\bar{\mathfrak{A}}}$-MMP$/U$, and $(h_i)_*\bar{\mathfrak{A}} = \mathfrak{A}_i$. In particular, each h_i is $K_{\bar{\mathfrak{A}}}$-negative.

Proof. Write $\mathfrak{A} = (X, \mathcal{F}, B, \mathbf{M}, t)$ and let $g : \widetilde{X} \to X$ be a foliated log resolution of $\mathfrak{A}$, such that the induced birational maps $g_i : \widetilde{X} \dashrightarrow X_i$ are morphisms. Let $0 < \delta \ll 1$ be a real number such that

$$K_{g_*^{-1}\mathfrak{A}} + (1 - 2\delta)(\operatorname{Exc}(g)^{\mathrm{ninv}} + (1 - t)\operatorname{Exc}(g)^{\mathrm{inv}}) \geq K_{g^*\mathfrak{A}}$$

and let $\widetilde{\mathfrak{A}} := (g_*^{-1}\mathfrak{A}, (1-\delta)(\operatorname{Exc}(g)^{\mathrm{ninv}} + (1-t)\operatorname{Exc}(g)^{\mathrm{inv}})$. Then $\widetilde{\mathfrak{A}}$ is klt. Let $f : \bar{\mathfrak{A}}' \to \widetilde{\mathfrak{A}}$ be a $\mathbb{Q}$-factorial terminalization of $\widetilde{\mathfrak{A}}$ with ambient variety $\overline{X}$, whose existence is guaranteed by Definition-Theorem 2.15, and let $\bar{\mathfrak{A}} := (\bar{\mathfrak{A}}', \epsilon \operatorname{Exc}(f))$ for some $0 < \epsilon \ll 1$.

We show that $\bar{\mathfrak{A}}$ and $h = f \circ g$ satisfy our requirements. By Lemma 2.14, $\bar{\mathfrak{A}}$ is terminal. By our construction, ϕ_i is $K_{\mathfrak{A}}$-negative. Since h is $K_{\bar{\mathfrak{A}}}$-negative, h_i is $K_{\bar{\mathfrak{A}}}$-negative. Therefore we have

$$K_{\bar{\mathfrak{A}}} = h_i^* K_{\mathfrak{A}_i} + E_i$$

for some $E_i \geq 0$ that is exceptional$/X_i$ and $\operatorname{Supp} E_i = \operatorname{Exc}(h_i)$. By [CHLMSSX25, Theorem 2.1.1], we may run a $K_{\bar{\mathfrak{A}}}$-MMP$/X_i$

$$\bar{\mathfrak{A}} \dashrightarrow \mathfrak{A}_{i,1} \dashrightarrow \mathfrak{A}_{i,2} \dashrightarrow \cdots \dashrightarrow \mathfrak{A}_{i,j} \dashrightarrow \dots$$

with scaling of ample divisor which either terminates, or the limit of the scaling numbers is 0. In either case, there exists j_i such that $K_{\mathfrak{A}_{i,j_i}}$ is a limit of movable$/X_i$ $\mathbb{R}$-divisors. Since $K_{\mathfrak{A}_{i,j_i}} \sim_{\mathbb{R},X_i} E_{i,j_i}$ where E_{i,j_i} is the image of E_i on X_{i,j_i}, by [Bir12, Lemma 3.3], we have $E_{i,j_i} = 0$. Thus $X_{i,j_i} = X_i$. Finally, note that

$$(h_i)_*\bar{\mathfrak{A}} = (\phi_i)_*\mathfrak{A} = \mathfrak{A}_i.$$

We conclude the proof. □

2.5. **Finiteness of weak lc models.**

Definition 2.19. Let $\pi : X \to U$ be a projective morphism between normal quasi-projective varieties. A *tuple* of nef$/U$ $\boldsymbol{b}$-Cartier $\boldsymbol{b}$-divisors on X is of the form $\mathfrak{M} := (\mathbf{M}_1, \dots, \mathbf{M}_n)$ for some positive integer n where each $\mathbf{M}_i$ is a nef$/U$ $\boldsymbol{b}$-Cartier $\boldsymbol{b}$-divisor. We define $\dim \mathfrak{M} := n$. We define

$$\operatorname{Span}_{\mathbb{R}}(\mathfrak{M}) := \left\{ \sum_{i=1}^{n} a_i \mathbf{M}_i \middle| a_i \in \mathbb{R} \right\}, \operatorname{Span}_{\mathbb{R}_{\geq 0}}(\mathfrak{M}) := \left\{ \sum_{i=1}^{n} a_i \mathbf{M}_i \middle| a_i \in \mathbb{R}_{\geq 0} \right\}.$$

For any nef$/U$ $\boldsymbol{b}$-Cartier $\boldsymbol{b}$-divisor $\mathbf{N}$, we define $(\mathfrak{M},\mathbf{N}) := (\mathbf{M}_1,\dots,\mathbf{M}_n,\mathbf{N})$.

Given a tuple $\mathfrak{M}$ of nef$/U$ $\boldsymbol{b}$-Cartier $\boldsymbol{b}$-divisors on X, let $V\subset \mathrm{Weil}_{\mathbb{R}}(X)\times \mathrm{Span}_{\mathbb{R}}(\mathfrak{M})$ be a finite dimensional affine subspace, $\mathcal{F}$ a foliation on X, and $t\in[0,1]$ a real number. We define:

$$V_{(X,\mathcal{F},t)} := \{\mathfrak{A}\mid \mathfrak{A}=(X,\mathcal{F},B,\mathbf{M},t),(B,\mathbf{M})\in V\}$$
$$\mathcal{L}(V_{(X,\mathcal{F},t)}) := \{\mathfrak{A}\mid \mathfrak{A}=(X,\mathcal{F},B,\mathbf{M},t)\in V_{(X,\mathcal{F},t)},\mathfrak{A}\text{ is lc},\mathbf{M}\in \mathrm{Span}_{\mathbb{R}_{\geq 0}}(\mathfrak{M})\}.$$
$$\mathcal{E}_\pi(V_{(X,\mathcal{F},t)}) := \{\mathfrak{A}\mid \mathfrak{A}\in \mathcal{L}(V_{(X,\mathcal{F},t)}),K_{\mathfrak{A}}\text{ is pseudo-effective}/U\}.$$

Lemma 2.20. *Let $\mathfrak{A}/U := (X,\mathcal{F},B,\mathbf{M},t)/U$ be a $\mathbb{Q}$-factorial klt algebraically integrable adjoint foliated structure such that $B+\mathbf{M}_X$ is big$/U$. Then there are finitely many $K_{\mathfrak{A}}$-non-positive extremal faces of $\overline{NE}(X/U)$. Moreover, all these faces are contractible.*

Proof. By [CHLMSSX25, Lemma 3.26] there exists an ample $\mathbb{R}$-divisor A on X and a klt algebraically integrable adjoint foliated structure $\mathfrak{A}'/U := (X,\mathcal{F},B',\mathbf{M}',t)/U$ such that $K_{\mathfrak{A}}\sim_{\mathbb{R},U} K_{\mathfrak{A}'}+A$. Thus any $K_{\mathfrak{A}}$-non-positive extremal face of $\overline{NE}(X/U)$ is a $(K_{\mathfrak{A}'}+\frac{1}{2}A)$-negative extremal face over U. By [CHLMSSX24, Theorem 1.3], there are only finitely many $(K_{\mathfrak{A}'}+\frac{1}{2}A)$-negative extremal rays over U, hence there are only finitely many $(K_{\mathfrak{A}'}+\frac{1}{2}A)$-negative extremal faces over U.

These faces are rational extremal faces by [CHLMSSX24, Theorem 1.3(4)], so each face has a supporting function$/U$ H such that $H-(K_{\mathfrak{A}'}+\frac{1}{2}A)$ is ample$/U$. The existence of the contraction follows from the base-point-freeness theorem [CHLMSSX25, Theorem 2.1.2]. □

Theorem 2.21 (Finiteness of weak lc models)**.** *Assume that*

- *$\pi\colon X\to U$ is a projective morphism between normal quasi-projective varieties,*
- *$\mathfrak{M}$ is a tuple of nef$/U$ $\boldsymbol{b}$-Cartier $\boldsymbol{b}$-divisors,*
- *V is a finite dimensional rational affine subspace of $\mathrm{Weil}_{\mathbb{R}}(X)\times\mathrm{Span}_{\mathbb{R}}(\mathfrak{M})$,*
- *$\mathcal{F}$ is an algebraically integrable foliation on X,*
- *$t\in[0,1]$ is a rational number.*
- *$\mathcal{C}\subset\mathcal{L}(V_{(X,\mathcal{F},t)})$ is a rational polytope, such that for any $\mathfrak{A}:=(X,\mathcal{F},B,\mathbf{M},t)\in\mathcal{C}$, $\mathfrak{A}$ is klt and $B+\mathbf{M}_X$ is big$/U$.*

Then there are finitely many birational maps$/U$ $\psi_j: X \dashrightarrow Z_j$, $1\leq j\leq l$ satisfying the following. For any $\mathfrak{A}\in\mathcal{C}$ and any weak lc model $\mathfrak{A}_Z/U$ of $\mathfrak{A}/U$ with induced birational map $\psi\colon X\dashrightarrow Z$, there exists an index $1\leq j\leq l$ such that $\psi_j\circ\psi^{-1}\colon Z\dashrightarrow Z_j$ is an isomorphism.

Proof. By [CHLMSSX25, Theorem 2.5.2], there are finitely many birational maps$/U$ $\phi_j: X\dashrightarrow Y_j$, $1\leq j\leq k$ satisfying the following.

(1) For any $\mathfrak{A}\in\mathcal{C}$ such that $K_{\mathfrak{A}}$ is pseudo-effective$/U$, there exists an index $1\leq j\leq k$ such that $(\phi_j)_*\mathfrak{A}/U$ is a $\mathbb{Q}$-factorial good minimal model of $\mathfrak{A}/U$.

(2) For any $\mathfrak{A}\in\mathcal{C}$ and any $\mathbb{Q}$-factorial minimal model $\mathfrak{A}_Y/U$ of $\mathfrak{A}/U$ with induced birational map $\phi\colon X\dashrightarrow Y$, there exists an index $1\leq j\leq k$ such that $\phi_j\circ\phi^{-1}\colon Y\dashrightarrow Y_j$ is an isomorphism.

Let
$$\mathcal{C}_j := \{(\phi_j)_*\mathfrak{A}\mid K_{(\phi_j)_*\mathfrak{A}}\text{ is nef}/U,\mathfrak{A}\in\mathcal{C}\}.$$
Then $\mathcal{C}_j\subset\mathcal{L}((V_j)_{(Y,(\phi_j)_*\mathcal{F},t)})$ for some $V_j\subset\mathrm{Weil}_{\mathbb{R}}(Y_j)\times\mathrm{Span}_{\mathbb{R}}(\mathfrak{M})$. Since nefness is a closed condition and $\mathcal{C}$ is a rational polytope, by [CHLMSSX25, Theorem 2.5.3], $\mathcal{C}_j$ is a rational polytope.

Now for any $\mathfrak{A}\in\mathcal{C}$ and any weak lc model $\mathfrak{A}_Z/U$ of $\mathfrak{A}/U$ with induced birational map $\psi\colon X \dashrightarrow Z$, we let

$$\mathcal{S}:=\{E\mid E \text{ is a prime divisor on } X \text{ that is exceptional}/Z, a(E,\mathfrak{A})=a(E,\mathfrak{A}_Z)\}.$$

By [CHLMSSX25, Theorem 2.2.3], there exists a birational morphism $g: Z'\to Z$ such that Z' is $\mathbb{Q}$-factorial and the divisors contracted by g are exactly the divisors in $\mathcal{S}$. Let $\mathfrak{A}_{Z'}:=g^*\mathfrak{A}_Z$, then $\mathfrak{A}_{Z'}/U$ is a $\mathbb{Q}$-factorial good minimal model of $\mathfrak{A}/U$, hence the induced birational map $\phi: X\dashrightarrow Z'$ is ϕ_j for some $1\le j\le k$. Now the contraction g is a contraction of a $K_{\mathfrak{A}_{Z'}}$-trivial extremal face F in $\overline{NE}(Y_j/Z)$ and $\mathfrak{A}_{Z'}\in\mathcal{C}_j$. By linearity of intersection numbers and since $K_{\mathfrak{A}_j}$ is nef$/U$ for any $\mathfrak{A}_j\in\mathcal{C}_j$, there exists a vertex $\mathfrak{A}_j^0$ of $\mathcal{C}_j$ such that F is a $K_{\mathfrak{A}_j^0}$-trivial extremal face$/U$.

There are only finitely many indices j. For each j, there are only finitely many vertices of $\mathcal{C}_j$. For each vertex $\mathfrak{A}_j^0$ of $\mathcal{C}_j$, by Lemma 2.20, there are only finitely many $K_{\mathfrak{A}_j^0}$-trivial extremal faces. Thus there are only finitely many possibilities of Z. □

3. Sarkisov program

Let $\mathfrak{A}$ and $\mathfrak{A}'$ be two outputs of two MMPs such that $f: X\to Z, f': X\to Z'$ are $K_{\mathfrak{A}}$ and $K_{\mathfrak{A}'}$-Mori fiber spaces respectively. This section we construct $\mathfrak{A}_i$ and $f_i: X_i\to Z_i$ by induction.

3.1. **Notation and set-up.** First set up a roof $\mathfrak{A}_W$ for two adjoint foliated structures $\mathfrak{A},\mathfrak{A}'$, such that after adding certain $\boldsymbol{b}$-divisors, they are also good minimal models of $\mathfrak{A}_W$.

Set-up 3.1. $\mathfrak{A}_W, W, \mathcal{F}_W, B_W, \mathbf{M}, t, U, g, h, \mathfrak{A}, X, \mathcal{F}, B, \mathfrak{A}', X', \mathcal{F}', B', f, f', L, R, \mathbf{L}, \mathbf{R}, \mathfrak{A}_W(\cdot,\cdot)$, $K_W(\cdot,\cdot)$ are defined in the following way.

1. $\mathfrak{A}_W/U=(W,\mathcal{F}_W,B_W,\mathbf{M},t)$ is a $\mathbb{Q}$-factorial terminal algebraically integrable adjoint foliated structure such that $\mathbf{M}$ is NQC$/U$ and descends to W, and $t\in\mathbb{Q}\cap[0,1)$.
2. $g: W\to X$ and $g': W\to X'$ are projective birational morphisms$/U$.
3. $\mathfrak{A}:=g_*\mathfrak{A}_W=:(X,\mathcal{F},B,\mathbf{M},t)$ and $\mathfrak{A}':=g'_*\mathfrak{A}_W=:(X',\mathcal{F}',B',\mathbf{M},t)$.
4. g and g' are $K_{\mathfrak{A}_W}$-negative and X,X' are $\mathbb{Q}$-factorial.
5. $f: X\to Z$ is a $K_{\mathfrak{A}}$-Mori fiber space$/U$ and $f': X'\to Z'$ is a $K_{\mathfrak{A}'}$-Mori fiber space$/U$.
6. L (resp. R) is an ample$/U$ $\mathbb{R}$-divisor on X (resp. X') such that $K_{\mathfrak{A}}+L\sim_{\mathbb{R}} f^*A_Z$ and $K_{\mathfrak{A}'}+R\sim_{\mathbb{R}} f'^*A_{Z'}$ for some ample$/U$ $\mathbb{R}$-divisors A_Z and $A_{Z'}$.
7. $\mathbf{L}:=\overline{L}$ and $\mathbf{R}:=\overline{R}$.
8. For any non-negative real numbers a,b, we denote by $\mathfrak{A}_W(a,b):=(\mathfrak{A}_W,a\mathbf{L}+b\mathbf{R})$ and $K_W(a,b):=K_{\mathfrak{A}_W(a,b)}$.

The trivial case is that $X\cong X'$. Otherwise, we shift (l,r) in $\mathfrak{A}_W(l,r)$ from $(l_0,r_0)=(1,0)$ to $(l_N,r_N)=(0,1)$. Precisely, we construct $1=l_0\ge l_1\ge\cdots\ge l_N=0$ and $0=r_0\le r_1\le\cdots\le r_N=1$. For each (l_i,r_i), we construct $h_i: W\dashrightarrow X_i$ by induction with the following conditions satisfied.

Condition 3.2. Notation and conditions as in Set-up 3.1. h_i, $X_i,\mathfrak{A}_i,\mathfrak{A}_i(\cdot,\cdot),K_i(\cdot,\cdot),f_i$, l_i,r_i,Z_i are as follows:

1. $0\le l_i\le 1$ and $0\le r_i\le 1$ are two real numbers.
2. $h_i: W\dashrightarrow X_i$ is a birational map$/U$.

(3) $\mathfrak{A}_i := (h_i)_*\mathfrak{A}_W$, $\mathfrak{A}_i(a,b) := (h_i)_*\mathfrak{A}_W(a,b)$, and $K_i(a,b) := (h_i)_*K_W(a,b)$ for any $a,b \geq 0$. In particular, $K_i(a,b) = K_{\mathfrak{A}_i(a,b)}$.
(4) $f_i : X_i \to Z_i$ is a $\mathbb{Q}$-factorial $K_{\mathfrak{A}_i}$-Mori fiber space$/U$.
(5) h_i is $K_W(l_i, r_i)$-non-positive and $\mathfrak{A}_i(l_i, r_i)$ is $\mathbb{Q}$-factorial klt. In particular, h_i does not extract any divisor.
(6) $K_i(l_i, r_i)$ is nef$/U$ and $K_i(l_i, r_i) \sim_{\mathbb{R},Z_i} 0$.

From (l_i, r_i) to (l_{i+1}, r_{i+1}), two Mori fiber spaces $X_i \dashrightarrow X_{i+1}$ are connected by a Sarkisov link, defined as follows.

Definition 3.3. Let $f_i : X_i \to Z_i$ and $f_{i+1} : X_{i+1} \to Z_{i+1}$ be two $\mathbb{Q}$-factorial Mori fiber spaces. A *Sarkisov link of Type* I (resp. II, III, IV) between $f_i : X_i \to Z_i$ and $f_{i+1} : X_{i+1} \to Z_{i+1}$ is of the following form:

$$\begin{tikzcd} V_i \arrow[r, dotted, "\phi_i"] \arrow[d, "p_i"'] & X_{i+1} \arrow[d, "f_{i+1}"] \\ X_i \arrow[d, "f_i"'] & Z_{i+1} \arrow[dl, "\beta_{i+1}"] \\ Z_i & \end{tikzcd}$$

I

$$\begin{tikzcd} V_i \arrow[r, dotted, "\phi_i"] \arrow[d, "p_i"'] & V_{i+1} \arrow[d, "q_{i+1}"] \\ X_i \arrow[d, "f_i"'] & X_{i+1} \arrow[d, "f_{i+1}"] \\ Z_i \arrow[r, "\cong"] & Z_{i+1} \end{tikzcd}$$

II

$$\begin{tikzcd} X_i \arrow[r, dotted, "\phi_i"] \arrow[d, "f_i"'] & V_{i+1} \arrow[d, "q_{i+1}"] \\ Z_i \arrow[dr, "\alpha_i"'] & X_{i+1} \arrow[d, "f_{i+1}"] \\ & Z_{i+1} \end{tikzcd}$$

III

$$\begin{tikzcd} X_i \arrow[rr, dotted, "\phi_i"] \arrow[d, "f_i"'] & & X_{i+1} \arrow[d, "f_{i+1}"] \\ Z_i \arrow[dr, "\alpha_i"'] & & Z_{i+1} \arrow[dl, "\beta_{i+1}"] \\ & T & \end{tikzcd}$$

IV

Here $p_i, q_{i+1}, \alpha_i, \beta_{i+1}$ are contractions of relative Picard number 1, p_i, q_{i+1} are birational, all diagrams commute, and ϕ_i is a sequence of flips with respect to certain adjoint foliated structure. Moreover, if all maps and morphisms in the commutative diagram are projective over a base U, then we call the Sarkisov links as Sarkisov links$/U$.

Suppose we have (l_i, r_i) and $h_i : W \dashrightarrow X_i$, then (l_{i+1}, r_{i+1}) and other notations are defined as follows.

Definition 3.4. Notation and conditions as in Set-up 3.1. Assume Condition 3.2 holds for some $i \in \mathbb{N}$. We define C_i, μ_i in the following way.
(1) C_i is a general f_i-vertical curve.
(2) μ_i is the unique positive real number such that $\mathbf{R}_{X_i} \equiv_{Z_i} \mu_i \mathbf{L}_{X_i}$. This is because $\rho(X_i/Z_i) = 1$, $\dim X_i > \dim Z_i$, and $\mathbf{R}_{X_i}, \mathbf{L}_{X_i}$ are big$/U$.
(3) Γ_i is the set of all real numbers λ satisfying the following.
  (a) $0 \leq \lambda \leq \frac{l_i}{\mu_i}$.
  (b) For any divisor $E \subset W$, we have
$$a(E, \mathfrak{A}_W(l_i - \mu_i\lambda, r_i + \lambda)) \leq a(E, \mathfrak{A}_i(l_i - \mu_i\lambda, r_i + \lambda)).$$
  (c) $K_i(l_i - \mu_i\lambda, r_i + \lambda)$ is nef$/U$.
(4) $s_i := \sup\{\gamma \mid \gamma \in \Gamma_i\}$.
(5) $l_{i+1} := l_i - s_i\mu_i$ and $r_{i+1} := r_i + s_i$.

Before constructing Sarkisov links, we have the following lemma about the (l_i, r_i).

Lemma 3.5. *Notation and conditions as in Set-up 3.1. Assume Condition 3.2 holds for some $i \in \mathbb{N}$. Adopt notations in Definition 3.4. Then we have:*
(1) Either $\Gamma_i = \{0\}$ or Γ_i is a closed interval, and $s_i \in \Gamma$.

(2) $l_{i+1} = l_i$ *if and only if* $r_{i+1} = r_i$ *if and only if* $s_{i+1} = 0$.
(3) $\mathfrak{A}_i(l_{i+1}, r_{i+1})$ *is klt.*
(4) $\Gamma \subset [0, 1 - r_i]$. *Moreover,* $r_{i+1} \leq 1$, *and if* $r_{i+1} = 1$, *then* $l_{i+1} = 0$.

Proof. It is obvious that $0 \in \Gamma$. The values of λ which satisfy each of the three conditions of Definition 3.4(4) form a closed and connected set in $\mathbb{R}_{\geq 0}$, and so is their intersection. This implies (1) and (2) follows from (1).

(3) By Lemma 2.17 we have $\mathfrak{A}_W(l_{i+1}, r_{i+1}) \geq \mathfrak{A}_i(l_{i+1}, r_{i+1})$, and $\mathfrak{A}_i(l_{i+1}, r_{i+1})$ is klt.

Assume that (4) does not hold, then $r_{i+1} > 1$, or $r_{i+1} = 1$ and $l_{i+1} > 0$. Then $\mathfrak{A}_W(l_{i+1}, r_{i+1})$ is terminal as $\mathbf{R}, \mathbf{L}$ descend to W. We have

$$0 \equiv_{Z_i} K_i(0,1) + l_{i+1}\mathbf{L}_{X_i} + (r_{i+1} - 1)\mathbf{R}_{X_i}.$$

Since $\mathbf{L}_{X_i}, \mathbf{R}_{X_i}$ are big$/U$, $\rho(Y_i/Z_i) = 1$, and $\dim X_i > \dim Z_i$, $\mathbf{L}_{X_i}, \mathbf{R}_{X_i}$ are ample$/Z_i$. Thus $K_i(0,1)$ is anti-ample$/Z_i$. In particular, $K_i(0,1)$ is not pseudo-effective. This is not possible as

$$K_W(0,1) = K_{\mathfrak{A}_W} + g'^*R \geq g'^*(K_{\mathfrak{A}'} + R) \sim_{\mathbb{R}} (f' \circ g')^* A_{Z'}$$

is pseudo-effective, $K_i(0,1) = (h_i)_* K_W(0,1)$, and h_i does not extract any divisor. This implies (4). $\square$

3.2. **Construction of Sarkisov link.** Notation and conditions as in Set-up 3.1. The induction starts at X. Let $h_0 := g, X_0 := X, f_0 := f, Z_0 = Z$ and $l_0 := 1, r_0 := 0$. Furthermore, let $\mathfrak{A}_0 := \mathfrak{A}$, $\mathfrak{A}_0(a,b) := g_*\mathfrak{A}_W(a,b)$, $K_0(a,b) := g_*K_W(a,b)$. Then Condition 3.2 holds for $i = 0$. We construct h_i, $X_i, \mathfrak{A}_i, \mathfrak{A}_i(\cdot,\cdot), K_i(\cdot,\cdot), f_i$, l_i, r_i, Z_i for any $i \in \mathbb{N}$ by induction in the following way.

If $s_i < \frac{l_i}{\mu_i}$, there are two cases corresponding to Definition 3.4(3.b) and (3.c) respectively.

Construction 3.6. Notation and conditions as in Set-up 3.1. Assume Condition 3.2 holds for some $i \in \mathbb{N}$. Adopt notations in Definition 3.4. Suppose that $s_i < \frac{l_i}{\mu_i}$. Assume there is a divisor $E \subset W$ such that for any $0 < \epsilon \ll 1$, we have $a(E, \mathfrak{A}_W(l_{i+1} - \mu_i\epsilon, r_{i+1} + \epsilon)) > a(E, \mathfrak{A}_i(l_{i+1} - \mu_i\epsilon, r_{i+1} + \epsilon))$ and $K_i(l_{i+1} - \mu_i\epsilon, r_{i+1} + \epsilon)$ is nef.

We construct $h_{i+1}, X_{i+1}, \mathfrak{A}_{i+1}, \mathfrak{A}_{i+1}(\cdot,\cdot), K_{i+1}(\cdot,\cdot), f_{i+1}, l_{i+1}, r_{i+1}, Z_{i+1}$ and p_i, V_i, ϕ_i, and possibly construct $\beta_{i+1}, q_{i+1}, V_{i+1}$ in the following way, so that Condition 3.2 holds for $i+1$, and $\mathfrak{A}_i(l_{i+1}, r_{i+1})$ and $\mathfrak{A}_{i+1}(l_{i+1}, r_{i+1})$ are crepant.

By our assumption, $K_i(l_{i+1}, r_{i+1})$ is nef$/U$. By Lemma 2.17, we have $\mathfrak{A}_W(l_{i+1}, r_{i+1}) \geq \mathfrak{A}_i(l_{i+1}, r_{i+1})$ and $\mathfrak{A}_i(l_{i+1}, r_{i+1})$ is klt. Since E lies on W and $a(E, \mathfrak{A}_W(l_{i+1} - \mu_i\epsilon, r_{i+1} + \epsilon)) \leq 0$ for any $\epsilon > 0$, it follows that $a(E, \mathfrak{A}_i(l_{i+1} - \mu_i\epsilon, r_{i+1} + \epsilon)) < 0$ for any $\epsilon > 0$, and hence

$$a(E, \mathfrak{A}_i(l_{i+1}, r_{i+1})) = \lim_{\epsilon \to 0^+} a(E, \mathfrak{A}_i(l_{i+1} - \mu_i\epsilon, r_{i+1} + \epsilon)) \leq 0.$$

By Lemma 3.5, $\mathfrak{A}_i(l_{i+1}, r_{i+1})$ is klt. By [CHLMSSX25, Theorem 2.2.3], there exists a projective birational morphism $p_i : V_i \to X_i$ such that $\mathrm{Exc}(p_i) = E$ and V_i is $\mathbb{Q}$-factorial. Since X_i is $\mathbb{Q}$-factorial, $\rho(V_i/X_i) = 1$. Pick $0 < \delta \ll \epsilon \ll 1$ and let

$$\widetilde{\mathfrak{A}}_i := \mathfrak{A}_i(l_{i+1} - \mu_i\epsilon - \delta, r_{i+1} + \epsilon).$$

Since $a(E, \mathfrak{A}_i(l_{i+1} - \mu_i\epsilon, r_{i+1} + \epsilon)) < 0$ and $0 < \delta \ll \epsilon$, $a(E, \widetilde{\mathfrak{A}}_i) < 0$. Let

$$\widetilde{\mathfrak{A}}_{V_i} := p_i^* \widetilde{\mathfrak{A}}_i.$$

Then $\widetilde{\mathfrak{A}}_{V_i}$ is klt. Moreover, we have

$$K_{\widetilde{\mathfrak{A}}_i} \equiv_{Z_i} -\delta \mathbf{L}_{X_{i+1}}$$

is anti-big$/Z_i$, hence $K_{\widetilde{\mathfrak{A}}_i}$ is not pseudo-effective$/Z_i$, and hence $K_{\widetilde{\mathfrak{A}}_{V_i}}$ is not pseudo-effective$/Z_i$. By [CHLMSSX25, Theorem 2.1.1], we may run a $K_{\widetilde{\mathfrak{A}}_{V_i}}$-MMP$/Z_i$ with scaling of an ample divisor which terminates with a Mori fiber space of $\widetilde{\mathfrak{A}}_{V_i}/Z_i$. Let $\mathfrak{A}_{V_i} := p_i^*\mathfrak{A}_i(l_{i+1}, r_{i+1})$, then $K_{\mathfrak{A}_{V_i}} \sim_{\mathbb{R},Z_i} 0$, so this MMP is $K_{\mathfrak{A}_{V_i}}$-trivial. There are two possibilities:

Case I. After finitely many flips $\phi_i : V_i \dashrightarrow X_{i+1}$, we obtain a Mori fiber space$/Z_i$ $f_{i+1} : X_{i+1} \to Z_{i+1}$. We have $\rho(X_{i+1}/Z_{i+1}) = 1$ and $\rho(X_{i+1}/Z_i) = 2$, so there exists a non-trivial contraction $\beta_{i+1} : Z_{i+1} \to Z_i$. Then we obtain a Sarkisov link$/U$ of Type I:

$$\begin{tikzcd}
V_i \arrow[r, dashed, "\phi_i"] \arrow[d, "p_i"'] & X_{i+1} \arrow[d, "f_{i+1}"] \\
X_i \arrow[d, "f_i"'] & Z_{i+1} \arrow[dl, "\beta_{i+1}"] \\
Z_i &
\end{tikzcd}$$

We let $h_{i+1} : W \dashrightarrow X_{i+1}$ be the induced birational map, $\mathfrak{A}_{i+1} := (h_{i+1})_*\mathfrak{A}_W$, $\mathfrak{A}_{i+1}(a,b) := (h_{i+1})_*\mathfrak{A}_W(a,b)$, and $K_{i+1}(a,b) := (h_{i+1})_*K_W(a,b)$ for any $a, b \geq 0$. Other notations are defined via Definition 3.4 for $i+1$.

Case II. After finitely many flips $\phi_i : V_i \dashrightarrow V_{i+1}$, we obtain a divisorial contraction $q_{i+1} : V_{i+1} \to X_{i+1}$. Then $(q_{i+1} \circ \phi_i)_*K_{\widetilde{\mathfrak{A}}_{V_i}}$ is not pseudo-effective$/Z_i$, so the induced contraction $f_{i+1} : X_{i+1} \to Z_{i+1} :\cong Z_i$ is a Mori fiber space. Then we obtain a Sarkisov link$/U$ of Type II:

$$\begin{tikzcd}
V_i \arrow[r, dashed, "\phi_i"] \arrow[d, "p_i"'] & V_{i+1} \arrow[d, "q_{i+1}"] \\
X_i \arrow[d, "f_i"'] & X_{i+1} \arrow[d, "f_{i+1}"] \\
Z_i \arrow[r, "\cong"] & Z_{i+1}
\end{tikzcd}$$

We let $h_{i+1} : W \dashrightarrow X_{i+1}$ be the induced birational map, $\mathfrak{A}_{i+1} := (h_{i+1})_*\mathfrak{A}_W$, $\mathfrak{A}_{i+1}(a,b) := (h_{i+1})_*\mathfrak{A}_W(a,b)$, and $K_{i+1}(a,b) := (h_{i+1})_*K_W(a,b)$ for any $a, b \geq 0$. Other notations are defined via Definition 3.4 for $i+1$.

Lemma 3.7. *Condition 3.2 holds for $i+1$ for both **Case I** and **Case II**.*

Proof. By Lemma 3.5(3), we only need to check Condition 3.2(4-6). By our construction, f_{i+1} is $K_{i+1}(l_{i+1}, r_{i+1})$-trivial, hence f_{i+1} is a $K_{\mathfrak{A}_{i+1}}$-Mori fiber space as $\mathbf{L}_{X_{i+1}}, \mathbf{R}_{X_{i+1}}$ are big. This implies Condition 3.2(4) for $i+1$.

Because h_i is $K_W(l_{i+1}, r_{i+1})$-non-positive, p_i is $K_{\mathfrak{A}_{V_i}}$-trivial, and the induced birational map $W_i \dashrightarrow V_i$ does not extract any divisor, it follows that h_{i+1} is also $K_W(l_{i+1}, r_{i+1})$-non-positive. Since $\mathfrak{A}_i(l_{i+1}, r_{i+1})$ is klt, $\mathfrak{A}_{V_i}$ and $\mathfrak{A}_{i+1}(l_{i+1}, r_{i+1})$ are klt. This implies Condition 3.2(5) for $i+1$.

$K_i(l_{i+1},r_{i+1})\sim_{\mathbb{R},Z_i} 0$ and $K_i(l_{i+1},r_{i+1})$ is nef$/U$, so $K_{\mathfrak{A}_{V_i}}\sim_{\mathbb{R},Z_i} 0$ and $K_{\mathfrak{A}_{V_i}}$ is nef$/U$, and so $K_{i+1}(l_{i+1},r_{i+1})\sim_{\mathbb{R},Z_i} 0$ and $K_{i+1}(l_{i+1},r_{i+1})$ is nef$/U$. This implies Condition 3.2(6) for $i+1$, and in particular, $\mathfrak{A}_i(l_{i+1},r_{i+1})$, $\mathfrak{A}_{V_i}$, $\mathfrak{A}_{i+1}(l_{i+1},r_{i+1})$ are crepant to each other. $\square$

The following lemma tracks the discrepancies through Sarkisov links of type II, which is necessary for showing termination of Sarkisov program.

Lemma 3.8. *Notation and conditions as in Construction 3.6. Assume that we are in* **Case II**. *Then for any $\delta',\epsilon'>0$, we have the following.*

(1) For any prime divisor F over X,

$$a(F,\mathfrak{A}_i(l_{i+1}-\mu_i\epsilon'-\delta',r_{i+1}+\epsilon'))\le a(F,\mathfrak{A}_{i+1}(l_{i+1}-\mu_i\epsilon'-\delta,r_{i+1}+\epsilon')).$$

(2) There exists a prime divisor F_0 over X, such that

$$a(F_0,\mathfrak{A}_i(l_{i+1}-\mu_i\epsilon'-\delta',r_{i+1}+\epsilon'))< a(F_0,\mathfrak{A}_{i+1}(l_{i+1}-\mu_i\epsilon'-\delta',r_{i+1}+\epsilon')).$$

Proof. By Construction 3.6, we know that for some $0<\delta\ll\epsilon<1$, $(q_i\circ\phi_i)$ is a sequence of steps of a $p_i^*K_i(l_{i+1}-\mu_i\epsilon-\delta,r_{i+1}+\epsilon)$-MMP$/Z_i$, and since this MMP contains a divisorial contraction, it is not trivial. We have

$$p_i^*K_i(l_{i+1}-\mu_i\epsilon-\delta,r_{i+1}+\epsilon)\sim_{\mathbb{R},Z_i}\frac{\delta}{\delta'}p_i^*K_i(l_{i+1}-\mu_i\epsilon'-\delta',r_{i+1}+\epsilon')$$

for any $\epsilon,\delta>0$. Therefore, $(q_{i+1}\circ\phi_i)$ is a sequence of steps of a $p_i^*K_i(l_{i+1}-\mu_i\epsilon'-\delta',r_{i+1}+\epsilon')$-MMP$/Z_i$, and since this MMP contains a divisorial contraction, it is not trivial. The lemma follows. $\square$

We need the following lemma for another construction.

Lemma 3.9. *Notation and conditions as in Set-up 3.1. Assume Condition 3.2 holds for some $i\in\mathbb{N}$. Adopt notations in Definition 3.4.*

Assume that $s_i<\frac{l_i}{\mu_i}$ and $\mathfrak{A}_W(l_{i+1}-\mu_i\epsilon,r_{i+1}+\epsilon)\ge\mathfrak{A}_i(l_{i+1}-\mu_i\epsilon,r_{i+1}+\epsilon)$ for any $0<\epsilon\ll 1$. Then $\mathfrak{A}_i(l_{i+1}-\mu_i\epsilon,r_{i+1}+\epsilon)$ is klt for any $0\le\epsilon\ll 1$, and there exists a contraction $\alpha_i:Z_i\to T_i$ such that $\rho(Z_i/T_i)=1$ and $K_i(l_{i+1}-\mu_i\epsilon,r_{i+1}+\epsilon)$ is not nef$/T_i$ for any $\epsilon>0$. In particular, $K_i(l_{i+1},r_{i+1})\sim_{\mathbb{R},T_i}0$.

Proof. For any $0\le\epsilon\ll 1$, since $\mathfrak{A}_W(l_{i+1}-\mu_i\epsilon,r_{i+1}+\epsilon)\ge\mathfrak{A}_i(l_{i+1}-\mu_i\epsilon,r_{i+1}+\epsilon)$ and $\mathfrak{A}_W(l_{i+1}-\mu_i\epsilon,r_{i+1}+\epsilon)$ is terminal, $\mathfrak{A}_i(l_{i+1}-\mu_i\epsilon,r_{i+1}+\epsilon)$ is klt. By our assumption, $K_i(l_{i+1}-\mu_i\epsilon,r_{i+1}+\epsilon)$ is not nef$/U$ for any $\epsilon>0$. By Lemma 2.20, there are finitely many $K_i(l_{i+1},r_{i+1})$-trivial extremal rays in $\overline{NE}(X_i/U)$, hence for any $0<\epsilon\ll 1$, there exists a $K_i(l_{i+1}-\mu_i\epsilon,r_{i+1}+\epsilon)$-negative extremal ray Σ_i that is $K_i(l_{i+1},r_{i+1})$-trivial. Note that we can identify $N_1(X_i/Z_i)$ as a subspace of $N_1(X_i/U_i)$ for each i. Let Σ_i' be the unique extremal ray of $\overline{NE}(X_i/Z_i)$, then Σ_i' is $K_i(l_{i+1}-\mu_i\epsilon,r_{i+1}+\epsilon)$-trivial for any ϵ, so $\Sigma_i'\neq\Sigma_i$, and so Σ_i' and Σ_i span a 2-dimensional extremal face F of $\overline{NE}(X_i/U)$. By Lemma 2.20, there exists a contraction $X_i\to T_i$ of F which factors through Z_i. We may let $\alpha_i:Z_i\to T_i$ be the induced contraction. $\square$

Construction 3.10. Notation and conditions as in Set-up 3.1. Assume Condition 3.2 holds for some $i\in\mathbb{N}$. Adopt notations in Definition 3.4. Suppose that $s_i<\frac{l_i}{\mu_i}$. Assume that for all $0<\epsilon\ll 1$, we have $K_i(l_{i+1}-\mu_i\epsilon,r_{i+1}+\epsilon)$ is not nef.

We construct h_{i+1}, $X_{i+1},\mathfrak{A}_{i+1},\mathfrak{A}_{i+1}(\cdot,\cdot),K_{i+1}(\cdot,\cdot),f_{i+1},l_{i+1},r_{i+1},Z_{i+1}$ and α_i,T_i,ϕ_i, and possibly construct $\beta_{i+1},q_{i+1},V_{i+1}$ in the following way, so that Condition 3.2 holds for $i+1$, and $\mathfrak{A}_i(l_{i+1},r_{i+1})$ and $\mathfrak{A}_{i+1}(l_{i+1},r_{i+1})$ are crepant.

By Lemma 3.9, there exists $\epsilon_0 > 0$, such that $\mathfrak{A}_i(l_{i+1} - \mu_i\epsilon, r_{i+1} + \epsilon)$ is klt for any $0 \leq \epsilon \leq \epsilon_0$, and there exists a contraction $\alpha_i : Z_i \to T_i$ such that $\rho(Z_i/T_i) = 1$ and $K_i(l_{i+1} - \mu_i\epsilon, r_{i+1} + \epsilon)$ is not nef$/T_i$ for any $0 < \epsilon \leq \epsilon_0$. We define α_i, T_i as above.

Pick $0 < \epsilon \ll \epsilon_0$. Since the generalized boundary of $\mathfrak{A}_W(l_{i+1} - \mu_i\epsilon, r_{i+1} + \epsilon)$ is big$/U$, the generalized boundary of $\mathfrak{A}_i(l_{i+1} - \mu_i\epsilon, r_{i+1} + \epsilon)$ is big$/U$. By [CHLMSSX25, Theorem 2.1.1], we may run a $K_i(l_{i+1} - \mu_i\epsilon, r_{i+1} + \epsilon)$-MMP$/T_i$ with scaling of an ample divisor which terminates. By Lemma 3.9, $K_i(l_{i+1}, r_{i+1}) \sim_{\mathbb{R},T_i} 0$, so this MMP is $K_i(l_{i+1}, r_{i+1})$-trivial.

There are three possibilities:

Case III. After finitely many steps of the MMP $\phi_i : X_i \dashrightarrow V_{i+1}$ we obtain a divisorial contraction$/T_i$ $q_{i+1} : V_{i+1} \to X_{i+1}$. Then $\rho(X_{i+1}/T_i) = 1$ and $(q_{i+1} \circ \phi_i)_* K_i(l_{i+1} - \mu_i\epsilon, r_{i+1} + \epsilon)$ is not pseudo-effective$/T_i$, so the induced contraction $f_{i+1} : X_{i+1} \to T_i$ is a Mori fiber space. Let $Z_{i+1} := T_i$, then we obtain a Sarkisov link$/U$ of Type III:

$$\begin{tikzcd}
X_i \arrow[r, dashrightarrow, "\phi_i"] \arrow[d, "f_i"'] & V_{i+1} \arrow[d, "q_{i+1}"] \\
Z_i \arrow[dr, "\alpha_i"'] & X_{i+1} \arrow[d, "f_{i+1}"] \\
 & Z_{i+1}
\end{tikzcd}$$

We let $h_{i+1} : W \dashrightarrow X_{i+1}$ be the induced birational map, $\mathfrak{A}_{i+1} := (h_{i+1})_*\mathfrak{A}_W$, $\mathfrak{A}_{i+1}(a,b) := (h_{i+1})_*\mathfrak{A}_W(a,b)$, and $K_{i+1}(a,b) := (h_{i+1})_*K_W(a,b) = K_{\mathfrak{A}_{i+1}(a,b)}$ for any $a, b \geq 0$. Other notations are defined via Definition 3.4 for $i+1$.

Case IV.1 After finitely many flips $\phi_i : X_i \dashrightarrow X_{i+1}$ we obtain a Mori fiber space$/T_i$ $f_{i+1} : X_{i+1} \to Z_{i+1}$. Let $\beta_{i+1} : Z_{i+1} \to T_i$ be the induced contraction. Then we obtain a Sarkisov link$/U$ of Type IV:

$$\begin{tikzcd}
X_i \arrow[rr, dashrightarrow, "\phi_i"] \arrow[d, "f_i"'] & & X_{i+1} \arrow[d, "f_{i+1}"] \\
Z_i \arrow[dr, "\alpha_i"'] & & Z_{i+1} \arrow[dl, "\beta_{i+1}"] \\
 & T_i &
\end{tikzcd}$$

We let $h_{i+1} : W \dashrightarrow X_{i+1}$ be the induced birational map, $\mathfrak{A}_{i+1} := (h_{i+1})_*\mathfrak{A}_W$, $\mathfrak{A}_{i+1}(a,b) := (h_{i+1})_*\mathfrak{A}_W(a,b)$, and $K_{i+1}(a,b) := (h_{i+1})_*K_W(a,b)$ for any $a, b \geq 0$. Other notations are defined via Definition 3.4 for $i+1$.

Case IV.2. After finitely many flips $\phi_i : X_i \dashrightarrow X_{i+1}$ we obtain a good minimal model $\mathfrak{A}_{i+1}(l_{i+1} - \mu_i\epsilon, r_{i+1} + \epsilon)/T_i$ of $\mathfrak{A}_i(l_{i+1} - \mu_i\epsilon, r_{i+1} + \epsilon)/T_i$. We let $h_{i+1} : W \dashrightarrow X_{i+1}$ be the induced birational map, $\mathfrak{A}_{i+1} := (h_{i+1})_*\mathfrak{A}_W$, $\mathfrak{A}_{i+1}(a,b) := (h_{i+1})_*\mathfrak{A}_W(a,b)$, and $K_{i+1}(a,b) := (h_{i+1})_*K_W(a,b)$ for any $a, b \geq 0$. Other notations are defined via Definition 3.4 for $i+1$.

Let C_i' be the strict transform of C_i on X_{i+1}. By the projection formula, we have

$$K_{i+1}(l_{i+1} - \mu_i\epsilon, r_{i+1} + \epsilon) \cdot C_i' = K_i(l_{i+1} - \mu_i\epsilon, r_{i+1} + \epsilon) \cdot C_i = 0.$$

We claim that $[C_i']$ is an extremal ray in $\overline{NE}(X_{i+1}/T_i)$. Suppose not, then we take generators θ, τ of two different extremal rays of $\overline{NE}(X_{i+1}/T_i)$, and we have $[C_i'] = a[\theta] + b[\tau]$ for some $a, b > 0$. Since $K_{i+1}(\epsilon)$ is nef$/T_i$, we have

$$K_{i+1}(l_{i+1} - \mu_i\epsilon, r_{i+1} + \epsilon) \cdot \theta = K_{i+1}(l_{i+1} - \mu_i\epsilon, r_{i+1} + \epsilon) \cdot \tau = 0,$$

which indicates that $K_{i+1}(l_{i+1} - \mu_i\epsilon, r_{i+1} + \epsilon) \sim_{\mathbb{R},T_i} 0$. However, since ϕ_i only consists of flips, $K_i(l_{i+1} - \mu_i\epsilon, r_{i+1} + \epsilon) \sim_{\mathbb{R},T_i} 0$, which is not possible.

Therefore, $[C_i']$ is an extremal ray in $\overline{NE}(X_{i+1}/T_i)$. By Lemma 3.9, $K_i(l_{i+1}, r_{i+1}) \sim_{\mathbb{R},T_i} 0$ and $K_i(l_{i+1}, r_{i+1})$ is nef$/U$. Since h_i is $K_W(l_{i+1}, r_{i+1})$-non-positive and ϕ_i is $K_i(l_{i+1}, r_{i+1})$-trivial, h_{i+1} is $K_W(l_{i+1}, r_{i+1})$-non-positive. By Lemma 3.9, $K_i(l_{i+1}, r_{i+1}) \sim_{\mathbb{R},T_i} 0$ and $K_i(l_{i+1}, r_{i+1})$ is nef$/U$, so $K_{i+1}(l_{i+1}, r_{i+1}) \sim_{\mathbb{R},T_i} 0$ and $K_{i+1}(l_{i+1}, r_{i+1})$ is nef$/U$. By Lemma 3.5, $\mathfrak{A}_i(l_{i+1}, r_{i+1})$ is klt, so $\mathfrak{A}_{i+1}(l_{i+1}, r_{i+1})$ is klt. Since $\mathbf{R}_{X_{i+1}}$ is big$/U$, by Lemma 2.20, there exists a contraction$/T_i$ $f_{i+1} : X_{i+1} \to Z_{i+1}$ of $[C_i']$. Let $\beta_{i+1} : Z_{i+1} \to T_i$ be the induced contraction, and we obtain a Sarkisov link$/U$ of Type IV:

$$\begin{tikzcd}
X_i \arrow[rr, dashed, "\phi_i"] \arrow[d, "f_i"'] & & X_{i+1} \arrow[d, "f_{i+1}"] \\
Z_i \arrow[dr, "\alpha_i"'] & & Z_{i+1} \arrow[dl, "\beta_{i+1}"] \\
& T_i &
\end{tikzcd}$$

Lemma 3.11. *Condition 3.2 holds for $i+1$ for **Case III** and **Case IV**.*

Proof. First check that Condition 3.2 holds for $i+1$ for **Case III** and **Case IV.1**. By Lemma 3.5(3), we only need to check Condition 3.2(4-6).

By our construction, f_{i+1} is a $K_{i+1}(l_{i+1} - \mu_i\epsilon, r_{i+1} + \epsilon)$-Mori fiber space, hence f_{i+1} is a $K_{\mathfrak{A}_{i+1}}$-Mori fiber space as $\mathbf{L}_{X_{i+1}}, \mathbf{R}_{X_{i+1}}$ are big. This implies Condition 3.2(4) for $i+1$.

Since h_i is $K_W(l_{i+1}, r_{i+1})$-non-positive and ϕ_i is $K_i(l_{i+1}, r_{i+1})$-trivial, h_{i+1} is $K_W(l_{i+1}, r_{i+1})$-non-positive. Since $\mathfrak{A}_i(l_{i+1}, r_{i+1})$ is klt, $\mathfrak{A}_{i+1}(l_{i+1}, r_{i+1})$ is klt. This implies Condition 3.2(5) for $i+1$.

By Lemma 3.9, $K_i(l_{i+1}, r_{i+1}) \sim_{\mathbb{R},T_i} 0$ and $K_i(l_{i+1}, r_{i+1})$ is nef$/U$, so $K_{i+1}(l_{i+1}, r_{i+1}) \sim_{\mathbb{R},T_i} 0$ and $K_{i+1}(l_{i+1}, r_{i+1})$ is nef$/U$. This implies Condition 3.2(6) for $i+1$.

Then we check that Condition 3.2 holds for $i+1$ for **Case IV.2**. By Lemma 3.5(3) and what we have discussed above, we only need to check Condition 3.2(4) for $i+1$. Since C_i can be any general vertical$/Z_i$ curve on X_i, f_{i+1} is a Mori fiber space. Since f_{i+1} is $K_{i+1}(l_{i+1}, r_{i+1})$-trivial and $\mathbf{L}_{X_{i+1}}, \mathbf{R}_{X_{i+1}}$ are big$/U$, f_{i+1} is $K_{\mathfrak{A}_{i+1}}$-negative, hence a $K_{\mathfrak{A}_{i+1}}$-Mori fiber space$/U$. This concludes Condition 3.2(4) for $i+1$.

In all cases, $\mathfrak{A}_i(l_{i+1}, r_{i+1})$ and $\mathfrak{A}_{i+1}(l_{i+1}, r_{i+1})$ are crepant because ϕ_i is $K_i(l_{i+1}, r_{i+1})$-trivial. □

The following lemmas track the discrepancies and μ_i through Sarkisov links of type III, IV, which are necessary for showing termination of Sarkisov program.

Lemma 3.12. *Notation and conditions as in Construction 3.10. Assume that we are in* **Case IV.1**. *Then $\mu_{i+1} < \mu_i$.*

Proof. We have

$$K_{\mathfrak{A}_{i+1}} + l_{i+1}\mathbf{L}_{X_{i+1}} + r_{i+1}\mathbf{R}_{X_{i+1}} = K_{i+1}(l_{i+1}, r_{i+1}) \sim_{\mathbb{R},Z_{i+1}} 0$$

and

$K_{\mathfrak{A}_{i+1}}+(l_{i+1}-\mu_i\epsilon)\mathbf{L}_{X_{i+1}}+(r_{i+1}+\epsilon)\mathbf{R}_{X_{i+1}}=K_{i+1}(l_{i+1}-\mu_i\epsilon,r_{i+1}+\epsilon)=(\phi_i)_*K_i(l_{i+1}-\mu_i\epsilon,r_{i+1}+\epsilon)$ is anti-ample$/Z_{i+1}$. Thus

$$\mu_i\mathbf{L}_{X_{i+1}}-\mathbf{R}_{X_{i+1}}\equiv_{Z_{i+1}}(\mu_i-\mu_{i+1})\mathbf{L}_{X_{i+1}}$$

is ample$/Z_{i+1}$, so $\mu_{i+1}<\mu_i$. □

Lemma 3.13. *Notation and conditions as in Construction 3.10. Assume that we are in* **Case IV.2**. *Then:*

(1) ϕ_i is not an isomorphism.

(2) $\mu_{i+1}=\mu_i$.

(3) For any prime divisor F over X and any $0<\epsilon\ll 1$,

$$a(F,\mathfrak{A}_i(l_{i+1}-\mu_i\epsilon,r_{i+1}+\epsilon))\leq a(F,\mathfrak{A}_{i+1}(l_{i+1}-\mu_i\epsilon,r_{i+1}+\epsilon)).$$

(4) There exists a prime divisor F_0 over X, such that

$$a(F_0,\mathfrak{A}_i(l_{i+1}-\mu_i\epsilon,r_{i+1}+\epsilon))<a(F_0,\mathfrak{A}_{i+1}(l_{i+1}-\mu_i\epsilon,r_{i+1}+\epsilon))$$

for any $0<\epsilon\ll 1$.

Proof. Since $K_i(l_{i+1}-\mu_i\epsilon,r_{i+1}+\epsilon)$ is not nef$/T_i$ but $(\phi_i)_*K_i(l_{i+1}-\mu_i\epsilon,r_{i+1}+\epsilon)$ is nef$/T_i$, we get (1).

Let $\psi_i:Y\to X_i$ and $\psi_{i+1}:Y\to X_{i+1}$ be common resolution and let C_Y be an irreducible curve on Y such that $\psi_i(C_Y)=C_i$. Denote $C'_{i+1}:=\psi_{i+1}(C_Y)$. Then

$$\begin{aligned}0&=K_i(l_{i+1}-\mu_i\epsilon,r_{i+1}+\epsilon)\cdot C_i=\psi_i^*K_i(l_{i+1}-\mu_i\epsilon,r_{i+1}+\epsilon)\cdot C_Y\\&=(\psi_{i+1})_*\psi_i^*K_i(l_{i+1}-\mu_i\epsilon,r_{i+1}+\epsilon)\cdot C'_{i+1}=(\phi_i)_*K_i(l_{i+1}-\mu_i\epsilon,r_{i+1}+\epsilon)\cdot C'_{i+1}\\&=(K_{i+1}(l_{i+1},r_{i+1})-\mu_i\epsilon\mathbf{L}_{X_{i+1}}+\epsilon\mathbf{R}_{X_{i+1}})\cdot C'_{i+1}=(-\mu_i\epsilon\mathbf{L}_{X_{i+1}}+\epsilon\mathbf{R}_{X_{i+1}})\cdot C'_{i+1}\\&=\epsilon(\mu_{i+1}-\mu_i)\mathbf{L}_{X_{i+1}}\cdot C'_{i+1},\end{aligned}$$

so $\mu_i=\mu_{i+1}$, which implies (2).

(3)(4) follow as ϕ_i is a sequence of steps of a $K_i(l_{i+1}-\mu_i\epsilon,r_{i+1}+\epsilon)$-MMP$/U$, and by (1), ϕ_i is not an isomorphism. □

Finally, we provide the condition under which the Sarkisov Program terminates.

Termination 3.14. Notation and conditions as in Set-up 3.1. Adopt notations in Definition 3.4.

If $s_i=\frac{l_i}{\mu_i}$, then we stop the induction and let $N=i+1$ and $X_N:=X_i,Z_N:=Z_i$ and $f_N:=f_i:X_N\to Z_N$. Note that $l_N=0$.

3.3. **Termination of Sarkisov program.** First we show that if the Sarkisov Program stops, then the output $X_N\to Z_N$ is $X'\to Z'$ as desired.

Lemma 3.15. *Notation and conditions as in Set-ups 3.1 and Termination 3.14. Then $l_N=0,r_N=1,f_N=f'$. In particular, $X_N=X'$ and $Z_N=Z'$.*

Proof. Since $K_N(0,r_N)$ is nef$/U$ and h_N is $K_W(0,r_N)$-non-positive, $K_W(0,r_N)$ is pseudo-effective$/U$, hence we have $g'_*K_W(0,r_N)\sim_{\mathbb{R},Z'}-(1-r_N)R$ is pseudo-effective$/Z'$. Thus $r_N=1$.

Let D be a sufficiently ample divisor on X_N and $\mathbf{D}:=\overline{D}$. Since $\mathfrak{A}_W$ is terminal and g' is $K_{\mathfrak{A}_W}$-negative, $\mathfrak{A}'$ is klt, hence $(\mathfrak{A}',\mathbf{R}+\epsilon\mathbf{D})$ is klt and g' is $(K_{\mathfrak{A}_W}+\mathbf{R}_W+\epsilon\mathbf{D}_W)$-negative for any $0<\epsilon\ll 1$. Since D is big$/U$, $\mathbf{D}_{X'}$ is big$/U$, hence $\mathbf{D}_{X'}$ is ample$/Z'$, and so $K_{\mathfrak{A}'}+R+\epsilon\mathbf{D}_{X'}$ is ample$/U$ for any $0<\epsilon\ll 1$. But h_N is also $(K_{\mathfrak{A}_W}+\mathbf{R}_W+\epsilon\mathbf{D}_W)$-negative, hence both X_N and X' are ample models$/U$ of $K_{\mathfrak{A}_W}+\mathbf{R}_W+\epsilon\mathbf{D}_W$. Thus $h_N=g'$, $X_N\cong X'$, and $K_N(0,1)=K_{\mathfrak{A}'}+R\sim_{\mathbb{R}} f'^*A_{Z'}$. Thus $K_N(0,1)$ is a supporting hyperplane of the unique extremal ray in $\overline{NE}(X'/Z')$. Since $K_N(0,1)\sim_{\mathbb{R},Z_N}0$ and $Z_N\neq X_N$, we have $f_N=f'$ and $Z_N=Z'$. $\square$

Then we show that the Sarkisov program terminates in finitely many steps.

Lemma 3.16. *Notation and conditions as in Set-ups 3.1. Assume Condition 3.2 holds for any $i\in\mathbb{N}$. Then there are only finitely many possibilities of h_i.*

Proof. $\mathfrak{A}_i(l_i,r_i)/U$ is a weak lc model of $\mathfrak{A}_W(l_i,r_i)/U$, so we are done by Theorem 2.21. $\square$

Lemma 3.17. *Notation and conditions as in Construction 3.6 and 3.10.*
Then there exists $i_0\in\mathbb{N}^+$, such that $l_i=l_{i_0}$ and $r_i=r_{i_0}$ for any $i\geq i_0$.

Proof. By Lemma 3.15, we may assume that $l_i>0$ for any i. By Lemma 3.16, there exists an infinite sequence of positive integers $i_0<i_1<\dots<i_k<\dots$ such that $h_{i_j}=h_{i_k}$ for any j,k. Thus $l_{i_j}=l_{i_k}$ by the construction of l_{i_j} and l_{i_k} in Constructions 3.6 and 3.10. By Constructions 3.6 and 3.10 again, we have that l_i is a decreasing sequence and r_i is an increasing sequence, so $l_i=l_{i_0}$ and $r_i=r_{i_0}$ for any $i\geq i_0$. $\square$

Theorem 3.18. *Notation and conditions as in Construction 3.6 and 3.10, after finitely many Sarkisov links, the Sarkisov Program terminates. That is, Termination 3.14 is eventually reached.*

Proof. Suppose it does not terminate and $l_i>0$ for all i. By Lemma 3.17, $l:=l_i$ and $r:=r_i$ are constants for $i\gg 0$. By Lemma 3.16, there exists an infinite sequence of positive integers $i_0<i_1<\dots<i_k<\dots$ such that $h_{i_j}=h_{i_k}$ for any j,k.

First, we suppose that ϕ_i is a Sarkisov link$/U$ of Type I or II for any $i\gg 0$. Since h_i does not extract any divisor, we have $\rho(W/U)\geq\rho(X_i/U)$ for any i. Therefore, for $i\gg 0$, ϕ_i is a Sarkisov link$/U$ of Type II. By Lemma 3.8, there exists $k>j$ and a prime divisor F_0 over X such that

$$a(F_0,\mathfrak{A}_{i_j}(l-\delta,r))<a(F_0,\mathfrak{A}_{i_k}(l-\delta,r))$$

for any $0<\delta\ll 1$. This is not possible as $\mathfrak{A}_{i_j}(l-\delta,r)=\mathfrak{A}_{i_k}(l-\delta,r)$.

Therefore, there exists an infinite sequence of positive integers $i_0'<i_1'<\dots<i_k'<\dots$ such that $\phi_{i_j'}$ is a Sarkisov link$/U$ of Type III or IV for any j. By Constructions 3.6 and 3.10, we have that $\mathfrak{A}_i(l,r)$ and $\mathfrak{A}_{i+1}(l,r)$ are crepant for any $i\gg 0$. Since ϕ_{i_j} is a Sarkisov link$/U$ of Type III or IV for any j, by Lemma 3.8 for Constructions 3.6 and Lemma 3.12, Lemma 3.13 for Construction 3.10, $\mathfrak{A}_{W,i_1}(l-\mu_{i_1}\epsilon,r+\epsilon)\geq\mathfrak{A}_{i_1}(l-\mu_i\epsilon,r+\epsilon)$ for any $0<\epsilon\ll 1$, hence $\mathfrak{A}_W(l-\mu_i\epsilon,r+\epsilon)\geq\mathfrak{A}_i(l-\mu_i\epsilon,r+\epsilon)$ for any $0<\epsilon\ll 1$ and any $i\geq i_1$, and so ϕ_i is a Sarkisov link$/U$ of Type III or IV for any $i\geq i_0'$. Since $\rho(X_i/U)>0$ for any i, ϕ_i is a Sarkisov link$/U$ of Type IV for any $i\gg 0$.

Since $h_{i_j}=h_{i_k}$ for any j,k, $\mu_{i_j}=\mu_{i_k}$ for any $j,k\gg 0$. By Lemmas 3.12 and 3.13(2), ϕ_i is a Sarkisov link$/U$ of Type IV.2 for any $i\gg 0$. By Lemma 3.8, there exists $k>j$ and a prime divisor F_0 over X such that

$$a(F_0,\mathfrak{A}_{i_j}(l-\mu_{i_j}\epsilon,r+\epsilon))<a(F_0,\mathfrak{A}_{i_k}(l-\mu_{i_k}\epsilon,r+\epsilon))$$

for any $0<\epsilon\ll 1$. This is not possible. $\square$

4. Proof of the main theorems

Proof of Theorem 1.3. Since ϕ_i is also a $((1-s)K_{\mathfrak{A}_W}+sK_W+H)$-MMP$/U$ for some $0<s\ll 1$ and ample $\mathbb{R}$-divisor H, by [CHLMSSX25, Lemma 3.29], after possibly replacing $\mathfrak{A}_W$, we may assume that $\mathbf{M}$ is NQC$/U$, $t\in[0,1)$, and $\mathfrak{A}$ is klt. By Lemma 2.18, we may assume that $\mathfrak{A}_W$ is $\mathbb{Q}$-factorial terminal, the nef part of $\mathfrak{A}_W$ descends to W, and g,g' are $K_{\mathfrak{A}_W}$-negative morphisms. Now we are in the setting of Set-up 3.1. Proceed the construction as in Section 3, we are done by Theorems 3.18 and 3.15. □

Proof of Theorem 1.4. It is a special case of Theorem 1.3. □

Proof of Theorem 1.1. $K_{\mathcal{F}}$ is not pseudo-effective, so it is algebraically integrable by [LLM23, Theorem 3.1] (essentially [CP19, Theorem 1.1]). Now Theorem 1.1 is a special case of Theorem 1.4. □

Proof of Theorem 1.2. It follows from Theorem 1.1 and [Mas24, Theorem 1.1]. (Note that [Mas24, Theorem 1.1] requires that the foliation is $\mathbb{Q}$-factorial F-dlt rather than only assuming that the ambient variety is $\mathbb{Q}$-factorial klt. On the other hand, any output of an MMP of a rank 2 foliated log smooth foliation on a threefold is $\mathbb{Q}$-factorial F-dlt.) □

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