FLOP BETWEEN ALGEBRAICALLY INTEGRABLE FOLIATIONS ON POTENTIALLY KLT VARIETIES

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Abstract. We prove that for any two minimal models of an lc algebraically integrable foliated triple on potentially klt varieties, there exist small birational models that are connected by a sequence of flops. In particular, any two minimal models of lc algebraically integrable foliated triples on $\mathbb{Q}$-factorial klt varieties are connected by a sequence of flops. We also discuss the connection between minimal models for possibly non-algebraically integrable foliations on threefolds, assuming the minimal model program for generalized foliated quadruples.

Contents

1. Introduction

We work over the field of complex numbers $\mathbb{C}$.

Flop between minimal models of varieties. Let X be a smooth projective variety with a pseudo-effective canonical divisor K_X. The minimal model program theory predicts that after a K_X-MMP, i.e., a sequence of K_X-divisorial contractions and flips, one obtains a minimal model $X_{\min}$ of X, such that $X_{\min}$ has $\mathbb{Q}$-factorial terminal singularities and $K_{X_{\min}}$ is nef, i.e., $K_{X_{\min}} \cdot C \geq 0$ for any curve C on $X_{\min}$. However, it is well known that for different K_X-MMPs, the minimal model $X_{\min}$ may not be unique. Therefore, it is interesting to ask what the connection is between different minimal models of X.

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Following the proof of the existence of the minimal model program for klt varieties [BCHM10], Kawamata proved that the induced birational map $X_1 \dashrightarrow X_2$ between any two minimal models X_1 and X_2 of a $\mathbb{Q}$-factorial terminal projective variety X can be decomposed into a sequence of K_{X_1}-flops [Kaw08, Theorem 1]. The same lines of the proof also hold when X is $\mathbb{Q}$-factorial klt. Later, following the proof of the existence of the minimal model program for lc varieties [Bir12, HX13, Has19], Hashizume improved this result to the case when X is $\mathbb{Q}$-factorial lc [Has20], and also considered the connection between minimal models of X when X is not $\mathbb{Q}$-factorial. More precisely, [Has20, Theorem 1.1] proved that for any lc projective variety X and two K_X-MMPs $X \dashrightarrow X_1$, $X \dashrightarrow X_2$, such that X_1 and X_2 are minimal models of X, there exist small birational morphisms $X_1' \to X_1$, $X_2' \to X_2$, and the induced birational map $X_1' \dashrightarrow X_2'$ can be decomposed into a sequence of $K_{X_1'}$-flops. Hashizume's result was later further improved to the case of NQC lc generalized pairs [Cha24], based on the proof of the existence of the minimal model program for such structures [HL23, LX23a, LX23b, Xie22].

Flop between minimal models of foliations. In recent years, there have been significant developments in the foundation of the minimal model program theory for foliations. For an lc foliation $\mathcal{F}$ on a klt variety X, the existence of a $K_{\mathcal{F}}$-minimal model program (cone theorem, contraction theorem, and the existence of flips) has been proven in dimension 2 [McQ08, Bru15], in dimension 3 when $\operatorname{rank} \mathcal{F} = 2$ and $\mathcal{F}$ is $\mathbb{Q}$-factorial F-dlt [Spi20, CS21, SS22], in dimension 3 when $\operatorname{rank} \mathcal{F} = 1$ and X is $\mathbb{Q}$-factorial [CS20], and when $\mathcal{F}$ is algebraically integrable [CHLX23, LMX24b, CHLMSSX24].

With all the aforementioned progress, it is interesting to ask whether two different minimal models of lc foliations on $\mathbb{Q}$-factorial klt varieties are connected by a sequence of flops, similar to the case of minimal models of varieties. [JV23, Theorem 1.1] provides a positive answer to this question for $\mathbb{Q}$-factorial F-dlt foliations of rank 2 on threefolds.

The goal of this paper is to provide a positive answer to this question for algebraically integrable foliations in any dimension. More precisely, we have the following:

Theorem 1.1. *Let $\mathcal{F}$ be an lc algebraically integrable foliation on a $\mathbb{Q}$-factorial klt projective variety X. Let $\phi_1 : (X, \mathcal{F}) \dashrightarrow (X_1, \mathcal{F}_1)$ and $\phi_2 : (X, \mathcal{F}) \dashrightarrow (X_2, \mathcal{F}_2)$ be two $K_{\mathcal{F}}$-MMPs, and let $\alpha : X_1 \dashrightarrow X_2$ be the induced birational map.*

Assume that $K_{\mathcal{F}}$ is pseudo-effective. Then α can be decomposed into a sequence of $K_{\mathcal{F}_1}$-flops.

Note that due to technical reasons, we need $(X_i, \mathcal{F}_i)$ to be minimal models that are obtained by running an MMP, which is slightly weaker than merely assuming that they are minimal models of $(X, \mathcal{F})$.

In practice, instead of considering $\mathcal{F}$ on a projective variety, it is more natural to consider foliated triples $(X, \mathcal{F}, B)$ associated with a projective morphism $X \to U$ between normal quasi-projective varieties. We have the following result:

Theorem 1.2. *Let $(X, \mathcal{F}, B)/U$ be a $\mathbb{Q}$-factorial lc algebraically integrable foliated triple such that X is klt. Let $\phi_1 : (X, \mathcal{F}, B) \dashrightarrow (X_1, \mathcal{F}_1, B_1)$ and $\phi_2 : (X, \mathcal{F}, B) \dashrightarrow (X_2, \mathcal{F}_2, B_2)$ be two $(K_{\mathcal{F}} + B)$-MMPs$/U$, and let $\alpha : X_1 \dashrightarrow X_2$ be the induced birational map$/U$.*

Assume that $K_{\mathcal{F}} + B$ is pseudo-effective$/U$. Then α can be decomposed into a sequence of $(K_{\mathcal{F}_1} + B_1)$-flops$/U$.

Inspired by [Has20, Theorem 1.2], we also would like to establish the connection between minimal models of lc algebraically integrable foliations on non-$\mathbb{Q}$-factorial varieties. We have the following result:

Theorem 1.3. *Let $(X,\mathcal{F},B)/U$ be an lc algebraically integrable foliated triple such that X is potentially klt. Let $\phi_1:(X,\mathcal{F},B)\dashrightarrow(X_1,\mathcal{F}_1,B_1)$ and $\phi_2:(X,\mathcal{F},B)\dashrightarrow(X_2,\mathcal{F}_2,B_2)$ be two $(K_{\mathcal{F}}+B)$-MMPs/U.*

Assume that $K_{\mathcal{F}}+B$ is pseudo-effective/U. Then there exist small $\mathbb{Q}$-factorial modifications $(X_1',\mathcal{F}_1',B_1')\to(X_1,\mathcal{F}_1,B_1)$ and $(X_2',\mathcal{F}_2',B_2')\to(X_2,\mathcal{F}_2,B_2)$, such that the induced birational map $\alpha':X_1'\dashrightarrow X_2'$ can be decomposed into a sequence of $(K_{\mathcal{F}_1'}+B_1')$-flops.

Remark 1.4. The same lines of the proof of our main theorems also apply to algebraically integrable generalized foliated quadruples [LLM23, Definition 1.2], possibly with the additional "NQC" condition. Furthermore, assuming the minimal model program for generalized foliated quadruples on $\mathbb{Q}$-factorial klt varieties in dimension 3, the same lines of the proof of our main theorems will also extend to (possibly non-algebraically integrable) foliations. In particular, this will provide an alternative proof to [JV23, Theorem 1.1]. See Section 4 for details.

Sketch of the proofs. For simplicity, let us assume that $U=\{pt\}$. First, we briefly recall Kawamata's original proof [Kaw08, Theorem 1] on the flop connection between $\mathbb{Q}$-factorial terminal pairs (X_1,B_1) and (X_2,B_2) associated with the birational map $\alpha:X_1\dashrightarrow X_2$: we take a general ample divisor $L_2\geq 0$ on X_2, consider its strict transform L_1 on X_1, and run a $(K_{X_1}+B_1+tL_1)$-MMP with scaling of an ample divisor for some $0<t\ll 1$. This MMP terminates with X_2 and is a sequence of $(K_{X_1}+B_1)$-flops due to three reasons:

(1) Since $0<t\ll 1$ and (X_1,B_1) is klt, (X_1,B_1+tL_1) is still klt.
(2) The bounded length of extremal rays ensures that this MMP is $(K_{X_1}+B_1)$-trivial.
(3) Since L_1 is big, this MMP terminates with a good minimal model by [BCHM10].

We want to follow the idea of [Kaw08, Theorem 1] to prove Theorem 1.2. With notations as in Theorem 1.2 and considering (1-3) above, we encounter the following three difficulties:

(i) Due to the failure of Bertini-type theorems, it is possible that $(X_2,\mathcal{F}_2,B_2+tL_2)$ is not lc for any ample $\mathbb{R}$-divisor $L_2\geq 0$ and any $t>0$.
(ii) Even if $(X_2,\mathcal{F}_2,B_2+tL_2)$ is lc, we also do not know how to prove that $(X_1,\mathcal{F}_1,B_1+tL_1)$ is lc for any $0<t\ll 1$.
(iii) Even if $(X_1,\mathcal{F}_1,B_1+tL_1)$ is lc, L_1 is only big. Unlike the usual klt pair case, we still do not know whether a $(K_{\mathcal{F}_1}+B_1+tL_1)$-MMP with scaling of an ample divisor will terminate with a good minimal model in general.

Of course, the lengths of extremal rays are still bounded, at least for algebraically integrable foliations [ACSS21, Theorem 3.9], [CHLX23, Theorem 2.2.1].

For issue (i), the natural idea is to consider the generalized foliated quadruple $(X_2,\mathcal{F}_2,B_2;t\mathbf{L}:=t\overline{L_2})$ instead of $(X_2,\mathcal{F}_2,B_2+tL_2)$, which is a standard way to bypass Bertini-type theorems since [LLM23]. For issue (iii), a key observation is that $K_{\mathcal{F}_2}+B_2+tL_2$ is ample. Since $\alpha:X_1\dashrightarrow X_2$ neither extracts nor contracts any divisors, $(X_2,\mathcal{F}_2,B_2,t\mathbf{L})$ is a good minimal model of $(X_1,\mathcal{F}_1,B_1,t\mathbf{L})$. Now, [LMX24b, Theorem 1.11] ensures the termination of the $(K_{\mathcal{F}_1}+B_1+t\mathbf{L}_{X_1})$-MMP with scaling of ample divisors, provided that $(X_1,\mathcal{F}_1,B_1,t\mathbf{L})$ is lc. Therefore, we are only left to resolve issue (ii); more precisely, we only need to show that $(X_1,\mathcal{F}_1,B_1,t\mathbf{L})$ is lc for some $t>0$.

The difficulty in issue (ii) arises from the fact that $(X_2,\mathcal{F}_2,B_2)$ has lc centers, and any small perturbation along the generic point of any lc center may destroy the lc structure.

A similar difficulty arose in [Has20] when considering flops connecting lc but not klt pairs (X_1, B_1) and (X_2, B_2). One key idea in [Has20, Proof of Theorem 1.1] is the observation that, over the generic point of any lc center of (X_1, B_1) or (X_2, B_2), the induced birational maps $X_1 \dashrightarrow X_2$ and $X_2 \dashrightarrow X_1$ are isomorphisms. In other words, there exist large open subsets $X_1^0 \subset X_1$ and $X_2^0 \subset X_2$ which contain all lc centers of (X_1, B_1) and (X_2, B_2), respectively, and the induced birational map $X_1^0 \dashrightarrow X_2^0$ is an isomorphism. In fact, [Has20, Section 4] provided counterexamples when no such large open subsets X_1^0 and X_2^0 exist, i.e., when the minimal models are not obtained as the outputs of the MMPs.

Unfortunately, the arguments in [Has20, Proof of Theorem 1.1] also do not work in our setting. This is because the lc centers of usual pairs form a proper closed subset of the ambient variety, but for any algebraically integrable foliation $\mathcal{F} \neq T_X$, the lc centers of $\mathcal{F}$ cover the entire variety X. Nevertheless, we have a way to resolve this issue: instead of only considering $(X_1, \mathcal{F}_1, B_1)$ and $(X_2, \mathcal{F}_2, B_2)$ and finding the corresponding open subsets X_1^0 and X_2^0, we aim to more concretely use the fact that $(X_1, \mathcal{F}_1, B_1)$ and $(X_2, \mathcal{F}_2, B_2)$ are outputs of MMPs.

More precisely, let $\phi_i : (X, \mathcal{F}, B) \dashrightarrow (X_i, \mathcal{F}_i, B_i)$ be $(K_{\mathcal{F}} + B)$-MMPs. Then we may take a high foliated log resolution $X' \to X$ of $(X, \mathcal{F}, B)$ so that the induced birational morphisms $X' \to X_i$ are also foliated log resolutions of $(X_i, \mathcal{F}_i, B_i)$. The MMPs $X \dashrightarrow X_i$ lift to MMPs $X' \dashrightarrow X_i'$ with induced $\mathbb{Q}$-factorial ACSS modifications $(X_i', \mathcal{F}_i, B_i') \to (X_i, \mathcal{F}_i, B_i)$. Here are the key observations:

- $X_2' \to X_2$ extracts only finitely many divisors E_j that are lc places of $(X_2, \mathcal{F}_2, B_2)$. We may choose L_2 so that it does not contain the generic point of the centers of any E_j.
- E_j are also the divisors extracted by $X_1' \to X_1$, and by a simple discrepancy comparison, α is an isomorphism near the generic point of any $\operatorname{center}_{X_1} E_j$. Thus, L_1 does not contain the generic point of any $\operatorname{center}_{X_1} E_j$. This implies that $\mathbf{L}_{X_1'}$ is the pullback of L_1.

Since $(X', \mathcal{F}', B', t\mathbf{L})$ is lc for any $t > 0$, $(X_1', \mathcal{F}_1', B_1', t\mathbf{L})$ is lc for any $0 < t \ll 1$. Since $\mathbf{L}_{X_1'}$ is the pullback of L_1, $(X_1, \mathcal{F}_1, B_1, t\mathbf{L})$ is lc. This resolves issue (ii), and the proof of Theorem 1.2 follows. Theorems 1.1 is an easy consequences of Theorem 1.2, while Theorem 1.3 is a consequence of Theorem 1.2 together with MMPs lifting to small $\mathbb{Q}$-factorial modifications (Proposition 3.1).

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2. Preliminaries

We will adopt the standard notation and definitions on MMP in [KM98, BCHM10] and use them freely. For foliations, foliated triples, generalized foliated quadruples, we adopt the notation and definitions in [LLM23, CHLX23] which generally align with [CS20, ACSS21, CS21] (for foliations and foliated triples) and [BZ16, HL23] (for generalized pairs and $\boldsymbol{b}$-divisors), possibly with minor differences.

2.1. **Foliations.**

Definition 2.1 (Foliations, cf. [ACSS21, CS21])**.** Let X be a normal variety. A *foliation* on X is a coherent sheaf $\mathcal{F}\subset T_X$ such that

(1) $\mathcal{F}$ is saturated in T_X, i.e. $T_X/\mathcal{F}$ is torsion free, and
(2) $\mathcal{F}$ is closed under the Lie bracket.

The *rank* of a foliation $\mathcal{F}$ on a variety X is the rank of $\mathcal{F}$ as a sheaf and is denoted by $\operatorname{rank}\mathcal{F}$. The *co-rank* of $\mathcal{F}$ is $\dim X-\operatorname{rank}\mathcal{F}$. The *canonical divisor* of $\mathcal{F}$ is a divisor $K_{\mathcal{F}}$ such that $\mathcal{O}_X(-K_{\mathcal{F}})\cong\det(\mathcal{F})$. If $\mathcal{F}=0$, then we say that $\mathcal{F}$ is a *foliation by points*.

Given any dominant map $h:Y\dashrightarrow X$ and a foliation $\mathcal{F}$ on X, we denote by $h^{-1}\mathcal{F}$ the *pullback* of $\mathcal{F}$ on Y as constructed in [Dru21, 3.2] and say that $h^{-1}\mathcal{F}$ is *induced by* $\mathcal{F}$. Given any birational map $g:X\dashrightarrow X'$, we denote by $g_*\mathcal{F}:=(g^{-1})^{-1}\mathcal{F}$ the *pushforward* of $\mathcal{F}$ on X' and also say that $g_*\mathcal{F}$ is *induced by* $\mathcal{F}$. We say that $\mathcal{F}$ is an *algebraically integrable foliation* if there exists a dominant map $f:X\dashrightarrow Z$ such that $\mathcal{F}=f^{-1}\mathcal{F}_Z$, where $\mathcal{F}_Z$ is the foliation by points on Z, and we say that $\mathcal{F}$ is *induced by* f.

A subvariety $S\subset X$ is called $\mathcal{F}$*-invariant* if for any open subset $U\subset X$ and any section $\partial\in H^0(U,\mathcal{F})$, we have $\partial(\mathcal{I}_{S\cap U})\subset\mathcal{I}_{S\cap U}$, where $\mathcal{I}_{S\cap U}$ denotes the ideal sheaf of $S\cap U$ in U. For any prime divisor P on X, we define $\epsilon_{\mathcal{F}}(P):=1$ if P is not $\mathcal{F}$-invariant and $\epsilon_{\mathcal{F}}(P):=0$ if P is $\mathcal{F}$-invariant. For any prime divisor E over X, we define $\epsilon_{\mathcal{F}}(E):=\epsilon_{\mathcal{F}_Y}(E)$ where $h:Y\dashrightarrow X$ is a birational map such that E is on Y and $\mathcal{F}_Y:=h^{-1}\mathcal{F}$.

Suppose that the foliation structure $\mathcal{F}$ on X is clear in the context. Then, given an $\mathbb{R}$-divisor $D=\sum_{i=1}^k a_iD_i$ where each D_i is a prime Weil divisor, we denote by $D^{\mathrm{ninv}}:=\sum\epsilon_{\mathcal{F}}(D_i)a_iD_i$ the *non-$\mathcal{F}$-invariant part* of D and $D^{\mathrm{inv}}:=D-D^{\mathrm{ninv}}$ the *$\mathcal{F}$-invariant part* of D.

Definition 2.2 (Singularities)**.** Let $(X,\mathcal{F},B)$ be a foliated triple. For any prime divisor E over X, let $f:Y\to X$ be a birational morphism such that E is on Y, and suppose that

$$K_{\mathcal{F}_Y}+B_Y:=f^*(K_{\mathcal{F}}+B)$$

where $\mathcal{F}_Y:=f^{-1}\mathcal{F}$. We define $a(E,\mathcal{F},B):=-\operatorname{mult}_E B_Y$ to be the *discrepancy* of E with respect to $(X,\mathcal{F},B)$. We say that $(X,\mathcal{F},B)$ is *lc* if $a(E,\mathcal{F},B)\geq-\epsilon_{\mathcal{F}}(E)$ for any prime divisor E over X. An *lc place* of $(X,\mathcal{F},B)$ is a prime divisor E over X such that $a(E,\mathcal{F},B)=-\epsilon_{\mathcal{F}}(E)$. An *lc center* of $(X,\mathcal{F},B)$ is the center of an lc place of $(X,\mathcal{F},B)$ on X.

Definition 2.3 (Potentially klt)**.** Let X be a normal quasi-projective variety. We say that X is *potentially klt* if (X,Δ) is klt for some $\mathbb{R}$-divisor $\Delta\geq 0$.

2.2. Foliated log smooth.

Definition 2.4 (cf. [ACSS21, 3.2 Log canonical foliated pairs], [CHLX23, Definition 6.2.1])**.** Let $(X,\mathcal{F},B)/U$ be an algebraically integrable foliated triple. We say that $(X,\mathcal{F},B)$ is *foliated log smooth* if there exists a contraction $f:X\to Z$ satisfying the following.

(1) X has at most quotient toric singularities.
(2) $\mathcal{F}$ is induced by f.
(3) (X,Σ_X) is toroidal for some reduced divisor Σ_X such that $\operatorname{Supp}B\subset\Sigma_X$. In particular, $(X,\operatorname{Supp}B)$ is toroidal, and X is $\mathbb{Q}$-factorial klt.
(4) There exists a log smooth pair (Z,Σ_Z) such that

$$f:(X,\Sigma_X)\to(Z,\Sigma_Z)$$

is an equidimensional toroidal contraction.

For any algebraically integrable foliated triple $(X,\mathcal{F},B)$, a *foliated log resolution* of $(X,\mathcal{F},B)$ is a birational morphism $h: X'\to X$ such that

$$(X',\mathcal{F}':=h^{-1}\mathcal{F},B':=h^{-1}_*B+\operatorname{Exc}(h))$$

is foliated log smooth, whose existence is guaranteed by [AK00, ACSS21, CHLX23].

Definition 2.5 (Foliated log smooth model, [LMX24b, Definition 4.10])**.** Let $(X,\mathcal{F},B)$ be an lc algebraically integrable foliated triple and $h: X'\to X$ a foliated log resolution of $(X,\mathcal{F},B)$. Let $\mathcal{F}':=h^{-1}\mathcal{F}$, and let $B'\geq 0$ and $E\geq 0$ be two $\mathbb{R}$-divisors on W satisfying the following.

(1) $K_{\mathcal{F}'}+B'=h^*(K_{\mathcal{F}}+B)+E$.
(2) $(X',\mathcal{F}',B')$ is foliated log smooth and lc.
(3) E is h-exceptional.
(4) For any h-exceptional prime divisor D such that
$$a(D,X,B)>-\epsilon_{\mathcal{F}}(E),$$
D is a component of E.

We say that $(X',\mathcal{F}',B')$ is a *foliated log smooth model* of $(X,\mathcal{F},B)$.

2.3. Models.

Definition 2.6 (Minimal models)**.** Let $(X,\mathcal{F},B)/U$ be a foliated triple and $\phi: X\dashrightarrow X'$ a birational map$/U$ which does not extract any divisor. Let $\mathcal{F}':=\phi_*\mathcal{F}$ and $B':=\phi_*B$.

We say that $(X',\mathcal{F}',B')/U$ is a *weak lc model* (resp. *minimal model*) of $(X,\mathcal{F},B)/U$ if $K_{\mathcal{F}'}+B'$ is nef$/U$, and for any prime divisor D on X which is exceptional over X', $a(D,\mathcal{F},B)\leq a(D,\mathcal{F}',B')$ (resp. $a(D,\mathcal{F},B)<a(D,\mathcal{F}',B')$). We say that $(X',\mathcal{F}',B')/U$ is a *semi-ample model* (resp. *good minimal model*) of $(X,\mathcal{F},B)/U$ if $(X',\mathcal{F}',B')/U$ is a weak lc model (resp. minimal model) of $(X,\mathcal{F},B)/U$ and $K_{\mathcal{F}'}+B'$ is semi-ample$/U$.

2.4. Relative Nakayama-Zariski decomposition.

Definition 2.7. Let $X\to U$ be a projective morphism between normal quasi-projective varieties and D an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. A birational map$/U$ $\phi: X\dashrightarrow X'$ is called a *D-MMP$/U$* if ϕ is a sequence of steps of a D-MMP$/U$, and either there exists a ϕ_*D-Mori fiber space$/U$ $X'\to Z$, or ϕ_*D is nef$/U$.

Definition 2.8. Let $\pi: X\to U$ be a projective morphism from a normal quasi-projective variety to a quasi-projective variety, D a pseudo-effective$/U$ $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X, and P a prime divisor on X. We define $\sigma_P(X/U,D)$ as in [LX23a, Definition 3.1] by considering $\sigma_P(X/U,D)$ as a number in $[0,+\infty)\cup\{+\infty\}$. We define $N_\sigma(X/U,D)=\sum_Q\sigma_Q(X/U,D)Q$ where the sum runs through all prime divisors on X and consider it as a formal sum of divisors with coefficients in $[0,+\infty)\cup\{+\infty\}$. We say that D is *movable$/U$* if $N_\sigma(X/U,D)=0$.

Lemma 2.9. *Let $X\to U$ be a projective morphism between normal quasi-projective varieties and let D be a pseudo-effective$/U$ $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. Let $\phi_i: X\dashrightarrow X_i$, $i\in\{1,2\}$ be two D-MMP$/U$ and let $D_i:=(\phi_i)_*D$ for $i\in\{1,2\}$. Then:*

(1) $\phi_2\circ\phi_1^{-1}$ is an isomorphism in codimension 1.
(2) D_1 and D_2 are crepant, i.e. for any birational morphisms $p_i: W\to X_i$, $i\in\{1,2\}$, we have $p_1^*D_1=p_2^*D_2$.

Proof. Since D be a pseudo-effective$/U$, D_1, D_2 are nef$/U$, hence D_1, D_2 are movable$/U$. By [LMX24b, Lemma 2.25], the divisors contracted by ϕ_i are exactly $\operatorname{Supp} N_\sigma(X/U, D)$. This implies (1). By (1), $p_1^*D_1 - p_2^*D_2$ is nef$/X_2$ and exceptional$/X_2$, and $p_2^*D_2 - p_1^*D_1$ is nef$/X_1$ and exceptional$/X_1$. By applying the negativity lemma twice, $p_1^*D_1 = p_2^*D_2$. This implies (2). □

2.5. **Generalized pairs and generalized foliated quadruples.**

Remark 2.10. For $\boldsymbol{b}$-divisors and generalized pairs, we will follow the notations and definitions in [BZ16, HL23]. For generalized foliated quadruples, we will follow [CHLX23].

Generalized pairs and generalized foliated quadruples are very technical concepts. To make the statements in this paper more concise, for most of the results we cite and prove, we refer only to the foliated triple version and briefly mention the detailed versions in Section 4. We refer the reader to [LMX24b, Appendix A] and [CHLX23] for results on generalized foliated quadruples.

The concepts defined above (singularities of foliations, foliated log smooth, foliated log smooth model, minimal models, etc.) can be similarly defined for generalized foliated quadruples. For the reader's convenience, we omit those technical definitions.

Definition 2.11 (NQC)**.** Let $X \to U$ be a projective morphism from a normal quasi-projective variety to a variety. Let D be a nef $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X and $\mathbf{M}$ a nef $\boldsymbol{b}$-divisor on X.

We say that D is NQC$/U$ if $D = \sum d_iD_i$, where each $d_i \geq 0$ and each D_i is a nef$/U$ Cartier divisor. We say that $\mathbf{M}$ is NQC$/U$ if $\mathbf{M} = \sum \mu_i\mathbf{M}_i$, where each $\mu_i \geq 0$ and each $\mathbf{M}_i$ is a nef$/U$ Cartier $\boldsymbol{b}$-divisor.

3. Proof of Main Theorem

Proof of Theorem 1.2. By Lemma 2.9, α is an isomorphism in codimension 1 and $\alpha_*B_1 = B_2$.

Let $h : X' \to X$ and $h_i : X' \to X_i$, $i \in \{1, 2\}$ be birational morphisms such that h_i is a foliated log resolution of $(X_i, \mathcal{F}_i, B_i)$ and h is a foliated log resolution of $(X, \mathcal{F}, B)$. Let $B' := h^{-1}_*B + \operatorname{Exc}(h)^{\mathrm{ninv}}$. Since $(X, \mathcal{F}, B)$ is lc, $(X', \mathcal{F}', B')$ is a foliated log smooth model of $(X, \mathcal{F}, B)$ and $B' = (h_i^{-1})_*B_i + F_i$ for some $F_i \geq 0$ that are exceptional$/X_i$.

By [LMX24b, Lemma 4.13], for $i \in \{1, 2\}$, we may run a $(K_{\mathcal{F}'} + B')$-MMP$/X_i$ with scaling of an ample divisor, which terminates with a good minimal model $(X_i', \mathcal{F}_i', B_i')/X_i$ of $(X', \mathcal{F}', B')/X_i$ with induced birational morphism $g_i : X_i' \to X_i$ and birational map $\phi_i' : X' \dashrightarrow X_i'$, such that $K_{\mathcal{F}_i'} + B_i' = g_i^*(K_{\mathcal{F}_i} + B_i)$. Since $K_{\mathcal{F}_i} + B_i$ is nef$/U$, $K_{\mathcal{F}_i'} + B_i'$ is nef$/U$. Thus $(X_i', \mathcal{F}_i', B_i')/U$ is a minimal model of $(X', \mathcal{F}', B')/U$.

Let $\alpha' : X_1' \dashrightarrow X_2'$ be the induced birational map$/U$. By Lemma 2.9, α' is an isomorphism in codimension 1. Therefore, the divisors extracted by g_1 are exactly the divisors extracted by g_2. Let $E_1, \dots, E_m$ be these divisors. Then each E_j is an lc place of $(X_i, \mathcal{F}_i, B_i)$ for $i \in \{1, 2\}$, so each E_j is also an lc place of $(X, \mathcal{F}, B)$ as

$$-\epsilon_{\mathcal{F}}(E_j) \leq a(E_j, \mathcal{F}, B) \leq a(E_j, \mathcal{F}_1, B_1) = -\epsilon_{\mathcal{F}}(E_j).$$

Therefore, ϕ_i is an isomorphism near $\operatorname{center}_X E_j$ for $i \in \{1, 2\}$ and any j, so α (resp. α^{-1}) is an isomorphism near $\operatorname{center}_{X_1} E_j$ (resp. $\operatorname{center}_{X_2} E_j$) for each j.

Let $L_2 \geq 0$ be an ample divisor on X_2 such that L_2 does not contain the generic point of $\operatorname{center}_{X_2} E_j$ for any j, and $\mathbf{L} := \overline{L_2}$. Since L is ample and $K_{\mathcal{F}_2} + B_2$ is nef$/U$, for any $s > 0$, X_2 is the ample model$/U$ of $K_{\mathcal{F}_2} + B_2 + s\mathbf{L}_{X_2}$.

Since $(X',\mathcal{F}',B')/U$ is lc and $\mathbf{L}$ descends to X', $(X',\mathcal{F}',B',s\mathbf{L})/U$ is lc (as a generalized foliated quadruple) for any $s>0$. For any $0<s\ll 1$, ϕ_1' is a sequence of steps of a $(K_{\mathcal{F}'}+B'+s\mathbf{L}_{X'})$-MMP$/U$, hence $(X_1',\mathcal{F}_1',B_1',s\mathbf{L})/U$ is lc. Since α (resp. α^{-1}) is an isomorphism near $\operatorname{center}_{X_1}E_j$ (resp. $\operatorname{center}_{X_2}E_j$) for each j, $\mathbf{L}_{X_1}=\alpha^{-1}_*L_2$ does not contain the generic point of $\operatorname{center}_{X_1}E_j$ for any j. Since X is $\mathbb{Q}$-factorial, X_1 is $\mathbb{Q}$-factorial. Thus

$$\mathbf{L}_{X_1'}=g_1^*\mathbf{L}_{X_1}.$$

Since $K_{\mathcal{F}_1'}+B_1'=g_1^*(K_{\mathcal{F}_1}+B_1)$, we have

$$K_{\mathcal{F}_1'}+B_1'+s\mathbf{L}_{X_1'}=g_1^*(K_{\mathcal{F}_1}+B_1+s\mathbf{L}_{X_1}),$$

hence $(X_1,\mathcal{F}_1,B_1,s\mathbf{L})/U$ is lc for any $0<s\ll 1$. We let $s_0<1$ be a positive real number such that $(X_1,\mathcal{F}_1,B_1,s\mathbf{L})/U$ is lc for any $0<s\leq s_0$.

Since h_1 is a foliated log resolution of $(X_1,\mathcal{F}_1,B_1)$ and $\mathbf{L}$ descends to X', h_1 is a foliated log resolution of $(X_1,\mathcal{F}_1,B_1,s\mathbf{L})$, and $(X',\mathcal{F}',B',s\mathbf{L})$ is a foliated log smooth model of $(X_1,\mathcal{F}_1,B_1,s\mathbf{L})$ for any $0<s\leq s_0$. By our construction, $K_{\mathcal{F}_2}+B_2+s\mathbf{L}_{X_2}$ is the ample model$/U$ of $K_{\mathcal{F}'}+B'+s\mathbf{L}_{X'}$ for any $0<s\leq s_0$. Therefore, $(X_2,\mathcal{F}_2,B_2,s\mathbf{L})/U$ is a semi-ample model of $(X',\mathcal{F}',B',s\mathbf{L})/U$ for any $0<s\leq s_0$. By [LMX24b, Lemma A.28] and since α does not extract any divisor, $(X_2,\mathcal{F}_2,B_2,s\mathbf{L})/U$ is a semi-ample model of $(X_1,\mathcal{F}_1,B_1,s\mathbf{L})/U$ for any $0<s\leq s_0$. Since α does not contract any divisor, $(X_2,\mathcal{F}_2,B_2,s\mathbf{L})/U$ is a good minimal model of $(X_1,\mathcal{F}_1,B_1,s\mathbf{L})/U$ for any $0<s\leq s_0$. In particular, $(X_1,\mathcal{F}_1,B_1,s\mathbf{L})/U$ has a good minimal model for any $0<s\leq s_0$.

By [LMX24b, Theorem 7.2], X_1 is $\mathbb{Q}$-factorial klt. By [LMX24b, Theorem A.13], for any $0<s\leq s_0$, we may run a $(K_{\mathcal{F}_1}+B_1+s\mathbf{L}_{X_1})$-MMP$/U$ which terminates with a good minimal model $(X_s,\mathcal{F}_s,B_s,s\mathbf{L})/U$ of $(X_1,\mathcal{F}_1,B_1,s\mathbf{L})/U$, such that X_s is $\mathbb{Q}$-factorial klt. Since $K_{\mathcal{F}_2}+B_2+s\mathbf{L}_{X_2}$ is ample$/U$, by [LMX24b, Lemma A.25], there exists an induced morphism $\psi_s:X_s\to X_2$. Since α is small and the induced birational map $\alpha_s:X_1\dashrightarrow X_s$ does not extract any divisor, ψ_s and α_s are small. However, since X_s is also $\mathbb{Q}$-factorial klt, ψ_s is the identity morphism, $\alpha_s=\alpha$, and $X_s=X_2$.

By [LMX24b, Theorem 1.12], $K_{\mathcal{F}_1}+B_1$ is NQC$/U$. Since $\alpha_s=\alpha$ can be decomposed into a sequence of steps of a

$$\left((K_{\mathcal{F}_1}+B_1+s_0\mathbf{L}_{X_1})+\left(\frac{s_0}{s}-1\right)(K_{\mathcal{F}_1}+B_1)\right)\text{-MMP}/U$$

for any $0<s\leq s_0$. By [LMX24b, Lemma B.6], for $0<s\gg s_0$, $\alpha=\alpha_s$ can be decomposed into a sequence of steps of a $(K_{\mathcal{F}_1}+B_1+s\mathbf{L}_{X_1})$-MMP$/U$, and each step is $(K_{\mathcal{F}_1}+B_1)$-trivial. In particular, $\alpha=\alpha_s$ can be decomposed into a sequence of $(K_{\mathcal{F}_1}+B_1)$-flops$/U$. □

Proof of Theorem 1.1. It is a special case of Theorem 1.2. □

To prove Theorem 1.3, we need the following proposition, Proposition 3.1, on MMP lifting. The difference between Proposition 3.1 and [LMX24b, Proposition 8.2] is that [LMX24b, Proposition 8.2] lifts the MMP to $\mathbb{Q}$-factorial ACSS models, while Proposition 3.1 lifts the MMP to small $\mathbb{Q}$-factorial modifications.

Proposition 3.1. *Let $(X,\mathcal{F},B)/U$ be an lc algebraically integrable foliated triple. Let*

$$(X,\mathcal{F},B):=(X_0,\mathcal{F}_0,B_0)\dashrightarrow(X_1,\mathcal{F}_1,B_1)\dashrightarrow\cdots\dashrightarrow(X_n,\mathcal{F}_n,B_n)\dashrightarrow\dots$$

be a (possibly infinite) sequence of $(K_{\mathcal{F}}+B)$-MMP$/U$. For each $i\geq 0$, we let $\psi_i:X_i\to T_i$ and $\psi_i^+:X_{i+1}\to T_i$ be the $(i+1)$-th step of this MMP and let $\phi_i:=(\psi_i^+)^{-1}\circ\psi_i:X_i\dashrightarrow$

X_{i+1} be the induced birational map. Let A be an ample/U $\mathbb{R}$-divisor on X and let A_i be the image of A on X_i for each i.

Assume that X is potentially klt. Let $h: Y \to X$ be a small $\mathbb{Q}$-factorial modifications of X, $\mathcal{F}_Y := h^{-1}\mathcal{F}$, and $B_Y := h^{-1}_*B$. Then there exist a (possibly infinite) sequence of birational maps

$$(Y, \mathcal{F}_Y, B_Y) := (Y_0, \mathcal{F}_{Y_0}, B_{Y_0}) \dashrightarrow (Y_1, \mathcal{F}_{Y_1}, B_{Y_1}) \dashrightarrow \cdots \dashrightarrow (Y_n, \mathcal{F}_{Y_n}, B_{Y_n}) \dashrightarrow \dots$$

satisfying the following. Let $\phi_{i,Y}: Y_i \dashrightarrow Y_{i+1}$ be the induced birational map. Then:

(1) For any $i \geq 0$, there exist a small $\mathbb{Q}$-factorial modifications $h_i: Y_i \to X_i$ such that $\mathcal{F}_{Y_i} = h_i^{-1}\mathcal{F}_i$, $B_{Y_i} = (h_i^{-1})_*B_i$, and $h_0 = h$.

(2) For any $i \geq 0$, $h_{i+1} \circ \phi_{i,Y} = \phi_i \circ h_i$.

(3) For any $i \geq 0$, $\phi_{i,Y}$ is a $(K_{\mathcal{F}_i} + B_{Y_i})$-MMP/$T_i$ and $(Y_{i+1}, \mathcal{F}_{Y_{i+1}}, B_{Y_{i+1}})/T_i$ is the output of this MMP, such that $\phi_{i,Y}$ is not the identity map.

(4) $\phi_{i,Y}: Y_i \dashrightarrow Y_{i+1}$ form a sequence of steps of a $(K_{\mathcal{F}_Y} + B_Y)$-MMP/$U$.

(5) Suppose that the $(K_{\mathcal{F}} + B)$-MMP/U above is an MMP/U with scaling of A. Let $A_Y := h^*A$ and let A_{Y_i} the image of A_Y on Y_i for each i. Let

$$\lambda_i := \inf\{t \geq 0 \mid K_{\mathcal{F}_i} + B_i + tA_i \text{ is nef}/U\}$$

be the $(i+1)$-th scaling number. Then:

(a) $\phi_{i,Y}$ is a sequence of steps of a $(K_{\mathcal{F}_{Y_i}} + B_{Y_i})$-MMP/$U$ with scaling of A_{Y_i}, and the scaling number of each step of $\phi_{i,Y}$ is λ_i.

(b) $\phi_{i,Y}: Y_i \dashrightarrow Y_{i+1}$ form a sequence of steps of a $(K_{\mathcal{F}_Y} + B_Y)$-MMP/$U$ with scaling of A_Y.

Proof. Since (4) follows from (3) and (5.b) follows from (5.a), we only need to prove (1)(2)(3) and (5.a).

Let n be a non-negative integer. We prove the proposition by induction on n and under the induction hypothesis that we have already constructed $(Y_i, \mathcal{F}_i, B_i)/U$ and h_i for any $i \leq n$ and $\phi_{i,Y}$ for any $i \leq n-1$ which satisfy (1)(2)(3)(5). When $n = 0$, this follows from our assumption, so we may assume that $n > 0$. We need to construct $\phi_{n,Y}, h_{n+1}$, and $(Y_{n+1}, \mathcal{F}_{n+1}, B_{n+1})/U$.

We let H_n be a supporting function of the extremal ray/U contracted by ψ_n and let

$$L_n := H_n - (K_{\mathcal{F}_n} + B_n),$$

such that $L_n = \lambda_n A_n$ if the $(K_{\mathcal{F}} + B)$-MMP/U with scaling of A. Then L_n is ample/T_n. Since $K_{\mathcal{F}_{Y_n}} + B_{Y_n} + h_n^*L_n \equiv_{T_n} 0$ and $K_{\mathcal{F}_{Y_n}} + B_{Y_n} + h_n^*L_n$ is nef/U, we may run a $(K_{\mathcal{F}_{Y_n}} + B_{Y_n})$-MMP/$T_n$ with scaling of an ample divisor, which is also a $(K_{\mathcal{F}_{Y_n}} + B_{Y_n})$-MMP/$T_n$ with scaling of $h_n^*L_n$, and is also a sequence of steps of a $(K_{\mathcal{F}_{Y_n}} + B_{Y_n})$-MMP/$U$ with scaling of $h_n^*L_n$. Since $(X_{n+1}, \mathcal{F}_{n+1}, B_{n+1})/T_n$ is a semi-ample model of $(Y_n, \mathcal{F}_{Y_n}, B_{Y_n})/T_n$, by [LMX24b, Theorem 1.11], the $(K_{\mathcal{F}_{Y_n}} + B_{Y_n})$-MMP/$T_n$ with scaling of an ample divisor terminates with a good minimal model $(Y_{n+1}, \mathcal{F}_{Y_{n+1}}, B_{Y_{n+1}})/T_n$ of $(Y_n, \mathcal{F}_{Y_n}, B_{Y_n})/T_n$. Since X_{n+1} is the ample model/T_n of $K_{\mathcal{F}_n} + B_n$, X_{n+1} is also the ample model/T_n of $K_{\mathcal{F}_{Y_{n+1}}} + B_{Y_{n+1}}$, so there exists an induced birational morphism $h_{n+1}: Y_{n+1} \to X_{n+1}$. Since $K_{Y_n} + B_{Y_n}$ is not nef/T_n and $K_{Y_{n+1}} + B_{Y_{n+1}}$ is nef/T_n, $\phi_{i,Y}$ is not the identity map.

Let $\phi_{n,Y}: Y_n \dashrightarrow Y_{n+1}$ be the induced birational map. By [LMX24b, Lemma 2.25], the divisors contracted by ϕ_n are exactly $\operatorname{Supp} N_\sigma(X_n/T_n, K_{\mathcal{F}_n} + B_n)$ and the divisors contracted by $\phi_{n,Y}$ are exactly $\operatorname{Supp} N_\sigma(Y_n/T_n, K_{\mathcal{F}_{Y_n}} + B_{Y_n})$. By [LX23a, Lemma

3.4(2)(3)],

$$\operatorname{Supp} N_\sigma(Y_n/T_n, K_{\mathcal{F}_{Y_n}} + B_{Y_n}) = \operatorname{Supp} N_\sigma(X_n/T_n, K_{\mathcal{F}_n} + B_n).$$

Thus h_{n+1} is small, which implies (1) for $n+1$. (2)(3) for $n+1$ follow immediately from our construction. Since $H_n \sim_{\mathbb{R},T_n} 0$, $\phi_{n,Y}$ is $(h_n^* H_n)$-trivial, so (5.a) for $n+1$ immediately follows. Thus (1)(2)(3) and (5.a) follow from induction on n and the proposition follows. □

Proof of Theorem 1.3. Let $h : X' \to X$ be a small $\mathbb{Q}$-factorial modification, $\mathcal{F}' := h^{-1}\mathcal{F}$, and $B' := h_*^{-1}B$. By Proposition 3.1, for $i \in \{1,2\}$, there exists a $(K_{\mathcal{F}'} + B')$-MMP$/U$ $\phi_i' : (X', \mathcal{F}', B') \dashrightarrow (X_i', \mathcal{F}_i', B_i')$ with induced small $\mathbb{Q}$-factorialization $h_i : X_i' \to X_i$, such that $\mathcal{F}_i' = h_i^{-1}\mathcal{F}_i$ and $B_i' = (h_i^{-1})_* B_i$. By Theorem 1.2, the induced birational map $\alpha' : X_1' \dashrightarrow X_2'$ can be decomposed into a sequence of $(K_{\mathcal{F}_1'} + B_1')$-flops$/U$. The theorem follows. □

4. Further discussions

4.1. **Generalized foliated quadruple versions.** The main theorems of our paper also hold for NQC lc algebraically integrable generalized foliated quadruples on $\mathbb{Q}$-factorial klt varieties. We cannot remove the NQC condition because the proof of Theorem 1.2 relies on Shokurov polytopes [LMX24b, Theorems 1.12], which also apply to NQC lc algebraically integrable generalized foliated quadruples [LMX24b, Theorem A.14], but no longer hold in the non-NQC case.

Definition 4.1. A *generalized foliated quadruple* $(X, \mathcal{F}, B, \mathbf{M})/U$ consists of a normal quasi-projective variety X, a foliation $\mathcal{F}$ on X, an $\mathbb{R}$-divisor $B \geq 0$ on X, a projective morphism $X \to U$, and a nef$/U$ $\boldsymbol{b}$-divisor $\mathbf{M}$, such that $K_{\mathcal{F}} + B + \mathbf{M}_X$ is $\mathbb{R}$-Cartier. We say that $(X, \mathcal{F}, B, \mathbf{M})/U$ is *NQC* if $\mathbf{M}$ is NQC$/U$.

Theorem 4.2 (Theorem 1.2)**.** *Let $(X, \mathcal{F}, B, \mathbf{M})/U$ be a $\mathbb{Q}$-factorial NQC lc algebraically integrable generalized foliated quadruple such that X is klt. Let $\phi_1 : (X, \mathcal{F}, B, \mathbf{M}) \dashrightarrow (X_1, \mathcal{F}_1, B_1, \mathbf{M})$ and $\phi_2 : (X, \mathcal{F}, B, \mathbf{M}) \dashrightarrow (X_2, \mathcal{F}_2, B_2, \mathbf{M})$ be two $(K_{\mathcal{F}} + B + \mathbf{M}_X)$-MMPs$/U$, and let $\alpha : X_1 \dashrightarrow X_2$ be the induced birational map$/U$.*

Assume that $K_{\mathcal{F}} + B + \mathbf{M}_X$ is pseudo-effective$/U$. Then α can be decomposed into a sequence of $(K_{\mathcal{F}_1} + B_1 + \mathbf{M}_{X_1})$-flops$/U$.

Proof. The proof follows from the same lines of the proof of Theorem 1.2 except that we replace [LMX24b, Lemma 4.13] with [LMX24b, Lemma A.29] and [LMX24b, Theorem 1.12] with [LMX24b, Theorem A.14] respectively. □

Theorem 4.3 (Theorem 1.3)**.** *Let $(X, \mathcal{F}, B, \mathbf{M})/U$ be an NQC lc algebraically integrable generalized foliated quadruple such that X is potentially klt. Let $\phi_1 : (X, \mathcal{F}, B, \mathbf{M}) \dashrightarrow (X_1, \mathcal{F}_1, B_1, \mathbf{M})$ and $\phi_2 : (X, \mathcal{F}, B, \mathbf{M}) \dashrightarrow (X_2, \mathcal{F}_2, B_2, \mathbf{M})$ be two $(K_{\mathcal{F}} + B + \mathbf{M}_X)$-MMPs$/U$.*

Assume that $K_{\mathcal{F}} + B + \mathbf{M}_X$ is pseudo-effective$/U$. Then there exist small $\mathbb{Q}$-factorial modifications $(X_1', \mathcal{F}_1', B_1', \mathbf{M}) \to (X_1, \mathcal{F}_1, B_1, \mathbf{M})$ and $(X_2', \mathcal{F}_2', B_2', \mathbf{M}) \to (X_2, \mathcal{F}_2, B_2, \mathbf{M})$, such that the induced birational map $\alpha' : X_1' \dashrightarrow X_2'$ can be decomposed into a sequence of $(K_{\mathcal{F}_1'} + B_1' + \mathbf{M}_{X_1'})$-flops.

Proof. The proof follows from the same lines of the proof of Theorem 1.3 except that we replace Theorem 1.2 with Theorem 4.2 and replace Proposition 3.1 with the corresponding generalized foliated quadruple version. The generalized foliated quadruple version of

Proposition 3.1 also from the same lines of the proof, except that we replace [LMX24b, Theorem 1.11] with [LMX24b, Theorem A.13]. □

4.2. **Non-algebraically integrable foliations on threefolds.** With the flop connection theorems for algebraically integrable foliations settled, it is natural to ask whether similar theorems hold for non-algebraically integrable foliations. Indeed, we can deduce similar theorems, provided that the minimal model program holds for generalized foliated quadruples. We first consider the following conjecture.

Conjecture 4.4. *Let $(X,\mathcal{F},B,\mathbf{M})/U$ be a $\mathbb{Q}$-factorial NQC lc generalized foliated quadruple such that $\dim X=3$ and X is klt. Let A be an ample$/U$ $\mathbb{R}$-divisor. Then:*

(1) The cone theorem (including the boundedness of length of extremal rays and the finiteness of $(K_{\mathcal{F}}+B+\mathbf{M}_X+A)$-negative extremal rays$/U$ in $\overline{NE}(X/U)$), contraction theorem, and the existence of flips hold for $(X,\mathcal{F},B,\mathbf{M})/U$.
(2) For any sequence of steps of a $(K_{\mathcal{F}}+B+\mathbf{M}_X)$-MMP$/U$ $(X,\mathcal{F},B,\mathbf{M}) \dashrightarrow (X',\mathcal{F}',B',\mathbf{M})$, X' is $\mathbb{Q}$-factorial klt.
(3) If $(X,\mathcal{F},B,\mathbf{M})/U$ has a minimal model, then we may run a $(K_{\mathcal{F}}+B+\mathbf{M}_X)$-MMP$/U$ with scaling of an ample$/U$ $\mathbb{R}$-divisor and any such MMP terminates with a minimal model of $(X,\mathcal{F},B,\mathbf{M})/U$.
(4) $(X,\mathcal{F},B+A,\mathbf{M})/U$ has a minimal model.

Now we consider the following variations of Conjecture 4.4:

Notation 4.5. We use the following subscripts to describe the special cases of Conjecture 4.4:

(1) "1": Conjecture 4.4 when $\operatorname{rank}\mathcal{F}=1$.
(2) "2": Conjecture 4.4 when $\operatorname{rank}\mathcal{F}=2$.
(3) "Fdlt": Conjecture 4.4 when $\operatorname{rank}\mathcal{F}=2$ and $(X,\mathcal{F},B,\mathbf{M})/U$ is F-dlt, with the extra conjecture that any for any sequence of steps of a $(K_{\mathcal{F}}+B+\mathbf{M}_X)$-MMP$/U$ $(X,\mathcal{F},B,\mathbf{M}) \dashrightarrow (X',\mathcal{F}',B',\mathbf{M})/U$, $(X',\mathcal{F}',B',\mathbf{M})$ is F-dlt.
(4) "Proj": Conjecture 4.4 when $U=\{pt\}$.
(5) "$\emptyset$": Conjecture 4.4.

We let Λ be the set of the following subscripts: $\emptyset$; 1; 2; Fdlt; Proj; 1, Proj; 2, Proj; Fdlt, Proj.

Theorem 4.6. *Let $\lambda\in\Lambda$ be a subscript. Assume Conjecture 4.4_λ holds.*

Let $(X,\mathcal{F},B)/U$ be a $\mathbb{Q}$-factorial lc foliated triple such that $\dim X=3$ and X is klt. Let $\phi_1:(X,\mathcal{F},B) \dashrightarrow (X_1,\mathcal{F}_1,B_1)$ and $\phi_2:(X,\mathcal{F},B) \dashrightarrow (X_2,\mathcal{F}_2,B_2)$ be two $(K_{\mathcal{F}}+B)$-MMPs$/U$, and let $\alpha: X_1 \dashrightarrow X_2$ be the induced birational map$/U$. Assume that $K_{\mathcal{F}}+B$ is pseudo-effective$/U$. Moreover:

(1) If λ contains 1 (resp. 2), then we assume that $\operatorname{rank}\mathcal{F}=1$ (resp. $\operatorname{rank}\mathcal{F}=2$).
(2) If λ contains Fdlt, then we assume that $\operatorname{rank}\mathcal{F}=2$ and $(X,\mathcal{F},B)$ is F-dlt.
(3) If λ contains Proj, then we assume that $U=\{pt\}$.

Then α can be decomposed into a sequence of $(K_{\mathcal{F}_1}+B_1)$-flops$/U$.

Sketch of the proof. The proof follows from the same lines of the proof of Theorem 1.2 with the following major modifications:

(1) When we take the foliated log resolution $h: X'\to X$, it shall be replaced with the foliated log resolution in the sense of [CS21, Definition 3.1] if $\operatorname{rank}\mathcal{F}=2$, whose

existence is guaranteed by [Can04], and in the sense of [LLM23, Definition 4.4] if $\operatorname{rank}\mathcal{F}=1$, whose existence is guaranteed by [MP13]. See also [LLM23, Theorem 4.5] for an explanation.

(2) The results [LMX24b, Lemma 4.13, A.25, A.28] used in the proof of Theorem 1.2 shall be replaced with the corresponding non-algebraically integrable version in dimension 3. However, all these results essentially only rely on the "very exceptional minimal model program", which in turn only relies on Birkar's general negativity lemma [Bir12, Lemma 3.3] (which always holds) and Conjecture 4.4_λ(3-4).

(3) The result [LMX24b, Theorem 7.2] used in the proof of Theorem 1.2 can be replaced with Conjecture 4.4_λ(2) (or, if λ contains Fdlt, with Notation 4.5(3)).

(4) The result [LMX24b, Theorem A.13] used in the proof of Theorem 1.2 can be replaced with Conjecture 4.4_λ(3).

(5) The results [LMX24b, Theorem 1.12, Lemma B.6] used in the proof of Theorem 1.2 essentially only rely on [HLS19, Lemma 5.3] (which always hold, or one can simply apply [LMX24a, Theorem 1.2]), the finiteness of $(K_{\mathcal{F}}+B+\mathbf{M}_X+A)$-negative extremal rays$/U$, and the boundedness of the length of extremal rays. The latter two conditions follow from Conjecture 4.4_λ(1). $\square$

In particular, Theorem 4.6(2) and Conjecture $4.4_{\mathrm{Fdlt,Proj}}$ together will provide a new proof of [JV23, Theorem 1.1]. We can also state a generalized foliated quadruple version of Theorem 4.6, and the proof will be similar. Again, due to technicalities, we omit the statements.

Postscript Remark. After we finished the first draft of this paper, we notice that [CM24, Theorem 8.1] proved the flop connection theorem for lc foliated triples $(X,\mathcal{F},B)/U$ such that $\dim X=3$, $\operatorname{rank}\mathcal{F}=2$, B is a $\mathbb{Q}$-divisor, (X,B) is klt, and $U=\{pt\}$. Their proof relies on the proof of a variation of Conjecture 4.4 when $\operatorname{rank}\mathcal{F}=2$, B is a $\mathbb{Q}$-divisor, $\mathbf{M}$ is a $\mathbb{Q}$-$\boldsymbol{b}$-divisor and $\boldsymbol{b}$-semi-ample, $(X,B,\mathbf{M})$ is klt, and $U=\{pt\}$. The proof follows from the same logic as the sketch of the proof of Theorem 4.6.

4.3. Final remarks.

Remark 4.7. In our main theorems, we require that X is $\mathbb{Q}$-factorial klt or potentially klt. This is because we only know the existence of the minimal model program for algebraically integrable foliations under these extra conditions [LMX24b, Theorems 1.2, 1.3], [CHLMSSX24, Theorem 1.11]. We expect the "potentially klt" condition to be removed if the existence of the minimal model program is proven for any lc algebraically integrable foliated triples.

Remark 4.8. It seems that we still do not know whether flops will connect minimal models for non-NQC lc generalized pairs, even when the ambient variety is $\mathbb{Q}$-factorial klt. For non-NQC $\mathbb{Q}$-factorial klt generalized pairs, although we cannot find a written proof, a positive answer can be deduced following the same lines of the proof in [Kaw08, Proof of Theorem 1].

Remark 4.9. The dual question to "flops connecting minimal models" is the Sarkisov program, which provides a way to connect Mori fiber spaces. The Sarkisov program was established in [HM13, Theorem 1.3] for $\mathbb{Q}$-factorial klt pairs, in [Liu21, Theorem 1.5] for $\mathbb{Q}$-factorial klt generalized pairs, and in [Mas24, Theorem 1.1] for $\mathbb{Q}$-factorial F-dlt foliated triples $(X,\mathcal{F},B)/U$ with $\dim X=3$, $\operatorname{rank}\mathcal{F}=2$, and $\lfloor B\rfloor=0$.

With the "flop connection part" settled in this paper, it is interesting to ask whether we can establish the Sarkisov program for lc algebraically integrable foliated triples on $\mathbb{Q}$-factorial klt varieties. This should follow from the log geography of minimal models, which is yet to be established for algebraically integrable foliations.

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